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ON THE FORMAL STRUCTURE OF THE PROPOSITIONAL CALCULUS II.

BY

MISS S. PANKAJAM, *University of Madras.*

[Received 17 March 1942]

In the previous paper with the same title, it was shown that with the accepted meanings of 'and', 'or', the totality of propositions form a distributive lattice P, in which the 'sum' and 'product' correspond to 'and', 'or'; we specifically examined the nature of negation with respect to these operations and showed that if we took the minimal meaning of negation, then the negation of any proposition turns out to be its product-complement in P. Lastly, it was shown that this meaning of negation is conformable to its meaning in Intuitionistic Logic (as for instance that formulated by Heyting*).

Further, implication was considered as a relation between propositions and identified with the lattice-order in P. However, the most characteristic feature of the propositional calculus is the acceptance of implication as an *operation* carried out on two propositions (so that if a, b are two propositions, then $a \supset b$ is also another proposition). The connection between these two view-points (viz., implication as a *relation* and as an *operation*) can be explained thus: the proposition 'a implies b' is the mere *consideration* of '$a \supset b$' while the relation 'a implies b' (which was identi-

* See my paper entitled 'On the Formal Structure of the Propositional Calculus I'. *Jour. Ind. Math. Soc.* (2) 5 (1941), 49-61. For the sake of brevity, this paper will hereafter be referred to as 'Formal Structure I'.

I wish to thank Dr. Vaidyanathaswamy for having kindly read my Mss. and offered valuable suggestions.

fied in the previous paper* with the order-relation $a < b$) is the *assertion* of '$a \supset b$'. In other words, the relation '$a < b$' holds if and only if $a \supset b = 1$, where $a \supset b$ is the proposition built up from a, b by the binary operation $\supset$. This supplies us with a first condition on the operation of implication, namely, $a \supset b = 1$ if and only if $a < b$ in the lattice-order.

It is the purpose of this paper to study from the point of view of lattice-algebra, the binary operation of implication $a \supset b$, and to shew that the proposition $a \supset b$ must be regarded as the product-complement of *a relative to b* in the propositional lattice P. The *product-complement* of a relative to b or (as it may be called) the *quotient* may be denoted by b/a or by (b, a); it is defined† as the element satisfying the following two conditions, viz.

D_1. $(b, a).\, a : < : b$.

D_2. *For all* t, a. $t < b$ *implies* $t < (b, a)$.

We can combine D_1 and D_2 and prove them equivalent to a single condition (D); namely

D. *The proposition* $t.\, a < b$ *is equivalent to* (*i.e. implies and is implied by*) *the proposition* $t < (b, a)$.

[For, $t.\, a < b : \supset : t < (b, a)$ follows from D_2. Again, $t < (b, a) : \supset : t.\, a < (b, a).\, a$, which by D_1 is $< b$, so that the reverse implication $t. < . (b, a) : \supset : t.\, a < b$ also holds. Thus, D_1. D_2 implies D. Conversely, it is obvious that D implies D_2 and also implies D_1; for, putting $t = (b, a)$ in the implication $t. < . (b, a) : \supset : t.\, a < D$, we get D_1.]

We observe that either of the conditions D_1 and D_2 or D alone, make the quotient (if it exists) unique. Also, since the negation of a proposition was seen‡ to be its

* See 'Formal Structure I', page 51.

† We call b the 'numerator' and a the 'denominator' of the quotient (b, a); we may notice that our definition of the relative product-complement is similiar to that of the product-complement. See p. 52, 'Formal Structure I'.

‡ cf. 'Formal Structure I'., Sec. 2.

product-complement, it follows that the binary operation of implication (corresponding to the relative product-complement) may be regarded as the generalization* of the unary operation of negation, thus bringing us back again from another point of view, to the vital connection between implication and negation.

Accordingly we conclude that in order to accept implication as a binary operation, we must assume that in P, every element possesses a product-complement relative to every other; thus the propositional lattice P must be such that not only do all product-complements exist but also all relative product-complements.

Since our argument proceeds on the basis of lattice-structure of propositions, it is clear that this view of implication as relative product-complement (or quotient) must be conformable to that of Intuitionistic Logic. This is explicitly shown in Sec. 5 of this paper.

N. B. A proposition substantially equivalent to this identification of implication with the quotient is also stated but not fully developed by Birkhoff† in his recent book on Lattice Theory.

1. *Implication as Quotient in the Lattice of Propositions.*

In the Introduction, we have already defined the quotient (b, a) in a distributive lattice; further, in order to identify implication with the quotient, we assume that the following plausible requirements are satisfied by the implication operation :

A_1. $a . \supset . \bar{b} : = : 1$ *if and only if* $a < b$.

A_2. *The principle of Inference.* $a. \ (a \supset b) : < : b$.

A_3. $a : \supset : b . \supset . c :: = :: a.b : \supset : c$.

* Thus if the element b relative to which the product-complement is taken is the element 0, then the relative product-complement becomes the ordinary product-complement.

† G. Birkhoff, Lattice Theory, *American Colloquium Publications.*

A_1 asserts the identity of the implication-relation with the lattice-order (as already considered in 'Formal Structure I') while A_2 is identical with the formal property of inference in any logical system; and finally A_3 is ordinarily satisfied by implication in most logical systems.

We now assert that A_1, A_2, A_3 together lead to the fundamental result, that *the proposition* $a \supset b$ *is 'the same as' the quotient* (b, a) *in* P. Since the quotient has already been shown to be unique, it follows that for the acceptance of implication as a binary operation, it is necessary and sufficient that the lattice P should admit not only product-complements but also quotients (i.e. product-complements of every element relative to every other). It should be noticed here that $<$ refers not only to lattice-order but also means asserted implication.

To prove that $a \supset b$ is the same as the quotient (b, a), we have to show that (in accordance with the definition D_1, D_2 of the quotient)

(i) $a. \ (a \supset b) : < : b.$

(ii) *If* $t.\ a : < : b$, *then* $t : < : a. \supset .\ b.$

Now, (i) is simply A_2; to prove (ii), we notice that $t.a < b$ is the assertion of $t.a \supset b$ which by A_3 is the same as $t : \supset : a \supset b$. But the assertion of $t : \supset : a. \supset . b$ is the same as the relation $t : < : a. \supset . b$ by A_1; thus (ii) is proved. We have thus established the fact that $a \supset b$ is the same as the quotient or the relative product-complement (b, a) in P.

Conversely, we can show that the relative-product-complement satisfies postulates similar to A_1—A_3, namely,

A (i). $(b, a) : = : 1$ if and only if $a < b$.

A (ii). $a. \ (b, a) : < : b.$

A (iii). $[(c, b), a)] = . \ (c, a. b).$

For, A(i), A(ii) follow from the definition (D_1, D_2) of the quotient. To prove A(iii), it is sufficient to shew

that the propositions $t : < : [(c, b), a]$ and $t < (c, a. b)$ are equivalent. Now by D, the first proposition is equivalent to $t\ a : < : (c, b)$ which again by D is equivalent to $t.\ ab < c$, which is equivalent to $t < (c, a. b)$. This proves A(iii).

As a direct consequence of A (iii), we have

THEOREM I. *If the numerator of a quotient is a normal element* (relative to c), then the quotient itself is a normal element relative to c.*

COROLLARY (i). $(b', a) = (ab)'$.

This is the particular case of Theorem I when $c = 0$.

It follows that (b', a) is a normal element of P whatever a may be, and since normal elements of the propositional lattice P are the negative propositions, we conclude that any implication in which the consequent is a negative proposition is itself a negative propos tion.

COROLLARY (ii). $(x', x) = x'$.

This is the special case of the previous corollary, for $a = b = x$. This property corresponds to the logical theorem "$x.\ \supset.$ not$-x$ is equivalent to not$-x$," i.e. to 'x is false'; this is valid both in intuitionistic as well as in classical logic.

An alternative form of Theorem I is the following.

If a is normal relative to b and b is normal relative to c, then a is normal relative to c.

For, since by hypothesis, $a = (b, \alpha)$ and $b = (c, \beta)$, it follows that $a = (b, \alpha) = [(c, \beta), \alpha] = (c, \alpha\beta)$ by A (iii) and hence is normal relative to c.

*An element is said to be normal relative to c if $[b, (b, c)] = b + c$. This is the generalization of the usual notion of normal elements. (*See* Formal Structure I, page 53); the condition for this is that b is of the form (c, k). We may observe that the theory of normal elements relative to b is the same as the theory of normal elements in P_b—the sub-lattice of elements $> b$.

The Canonical Form of a Quotient.

A quotient is said to be in the 'Canonical Form' if its numerator b is $<$ the denominator a. It is easy to see that (b, a) can be expressed in the canonical form in two ways, thus,

$$(b, a) = (b, a+b) = (ab, a).$$

For by D, the element (b, a) is such that the proposition $ta<b$ is equivalent to the proposition $t < (b, a)$, but $t. a < b$ is evidently equivalent both to $t(a+b)<b$ and to $ta<ab$, so that the elements (b, a), $(b, a+b)$, (ab, b) are all identical.

We next study the properties of quotients in the general lattice P assuming of course that all relative product-complements exist in P.

2. *Simple properties of Quotients in P.*

THEOREM 2. (i) $a'+b < (b, a)$.

(ii) *Syllogistic Principle.* $(c, b).\ (b, a) < (c, a)$.

(iii) $b.(y, b) = by$; *in particular,* $(y, 1) = y$.

(iv) *If the denominator (numerator) of a quotient increase, then the quotient decreases (increases).*

[We notice that (i) implies $b < (b, a)$ which gives a property common to both classical and Intuitionistic logic, viz. $b. \supset .(a \supset b)$ *or*, 'a true proposition is implied by any proposition.' Similiarly, (ii) gives the principle of syllogism while (iii) is a variant of the principle of inference, viz. $b(b \supset y) : \supset : y$.]

PROOF. (i) Since in P, $a(a'+b) : = : ab$ which is $<b$, it follows from D, that $a'+b < (b, a)$.

(ii) Now, the proposition $t < (c', b).(b, a)$ is equivalent to the proposition $\{t<(c, b)\}\cdot\{t<(b, a)\}$ which by D is equivalent to $(tb < c)(ta < b)$, this implies that $ta < tb < c$ which by D is equivalent to $t < (c, a)$. Thus, $t < (c, b) \times (b, a)$ implies $t < (c, a)$ so that we have

$$(c, b).\ (b, a) : < : (c, a).$$

(iii) Since by (i), $y < (y, b)$, it follows that $by :<: b(y, b)$; again since $b(y, b) :<: b$ and by D_1, $b. (y, b): <: y$, it follows that $b(y, b) :<: by$, hence our result.

(iv) Let $b<c$ in (ii) so that $(c, b) = 1$; then it reduces to $(b, a) < (c, a)$. Again putting $a < b$ in (ii), $(b, a) = 1$ and it reduces to $(c, b) < (c, a)$.

Corollary (1). $(b, a) < (a', b') < (b'', a'')$.

For, by **2** (ii), $b'. (b, a) = (0, b). (b, a) < (0, a) = a'$; hence, $(b, a) < (a', b')$. Repeating the same arguement, we have $(a', b') < (b'', a'')$.

Corollary (2). $(b', a) < (a', b)$.

For, by the previous corollary, $(b', a) < (a', b'')$. And since by a theorem* of product-complements $b'' > b$, it follows from **2** (ii), that $(a', b'') < (a', b)$ and hence the result.

Therefore, in Intuitionistic as well as in classical logic, the two propositions, viz.

(1) $a. \supset. b: \supset:$ not$-b. \supset.$ not$-a.$

(2) $a. \supset.$ not$-b: \supset: b. \supset.$ not$-$ $b.$

can always be asserted.

Corollary (3). $(b, a) = (a', b')$ for all b, a if and only if P is a Boolean algebra.

For, let P be a Boolean algebra so that $b = b''$, $a = a''$; we then deduce from Corollary (1) that $(b, a) < (a', b') < (b'' < a'') = (b, a)$. Hence, $(b, a) = (a', b')$, Conversely let $(b, a) = (a', b')$ for all a, b. Then to show that P is a Boolean algebra, put $a = 1$ in the above equality so that $(b, 1) = (0, b')$ or $b = b''$ for all b. Hence P is a Boolean algebra.

Theorem 3. $(x, x') = x''$.

* 'Formal structure I', Theorem 3.

By Theorem 2 (iv), $x'' = (0, x') < (x, x')$. And by Theorem 2, Cor. (1), and Theorem 1 Cor. (ii)

$$(x, x') < (x'', x') = x''.$$

From this theorem is seen the fact that if $(x, x') = x$ for all x, then $x = x''$ for all x so that *every element is normal and therefore every element is simple.** If P is the lattice of propositions, this means that the assumption 'not $-x$. ⊃. x is *equivalent* to x' makes the logic into classical logic.

As a matter of fact, the proposition x'. ⊃. x: ⊃: x is one of the three postulates of the Lukasiewikcz—Tarski system for classical Logik.†

THEOREM 4. (i) $(b, a) = a'$ *if* $b < a'$.

(ii) $(b, a) = (a' + b)$ *if and only if* a *is simple.*

PROOF. (i) Since $0 < b < a'$, we have by Theorem 2 (iv), $a' = (0, a) < (b, a) < (a', a) = a'$ by Theorem 1, Cor. (ii); hence $(b, a) = a'$.

(ii) Let a be simple so that $a + a' = 1$; then since $t < (b, a)$ implies $ta < b$ and since also $ta' < a'$, we have $t = ta + ta' < (a' + b)$. Thus $t < (b, a)$ implies $t < (a' + b)$ and hence $(b \ \ a) < (a' + b)$. But by Theorem 2 (i), $(b, a) > a' + b$; therefore it follows that if a is simple, then $(b, a) = a' + b$.

Next let $(b, a) = (a' + b)$ for every b. Then to show that a is simple, put $b = a$ in this; we have then, $1 = (a, a) = a' + a$ so that a is simple.

3. *The correspondence* $x \rightarrow (b, x)$.

Here, (b, x) is the product-complement of x relative to b and by Theorem 2 (i), is always $> (b + x') > b$; if we take the normal form $(b, b + x)$ for (b, x), it is clear that (b, x) is the product-complement of $b + x$ in the sub-

* 'Formal Structure I' Theorem 5.

† c.f. A Tarski 'Grundzuege der Systemen—Kalkuls' *Fundamenta Mathematica* 35 (1935).

lattice P_b of elements $>b$. Thus in dealing with quotients which have a fixed numerator b, we are really concerned with product-complements in the sub-lattice P_b. Accordingly, known properties of product-complements may be extended straightaway to quotients with a fixed numerator b.

THEOREM 5. *If* Σa *exists, then* $\prod_a (b, a)$ *exists and* $= (b, \Sigma a)$.

PROOF. For, since by hypothesis Σa exists, it follows that $\Sigma a + b$ exists, and since by definition of lattice-sum, $\Sigma a + b = \Sigma_a (a+b)$, we have $\Sigma_a (a+b)$ exists. Hence since $(b, a) = (b, a+b)$ is the product-complement of $a+b$ in the lattice P_b, we get from the theory of product-complements that $\prod_a (b, a+b) = \prod (b, a)$ exists and $= [b, \Sigma(b+a)]$ i.e. $= (b, \Sigma a)$.

THEOREM 6. (i) $(b+a) < [b, (b, a)]$.

(ii) $(b, a) = \{ b, [b, (b, a)] \}$.

These two results are the applications to P_b of two well-known properties* of product-complements, viz. (i) $a < a''$; (ii) $a' = a'''$.

COROLLARY. If P is a Boolean algebra, then P_b is also a Boolean algebra so that every element of P_b is a normal element. It follows in this case that $<$ can be replaced by $=$ in (i).

Conversely, let $(b+a) = [b, (b, a)]$ for all b, a; putting $b = 0$ in the above equality, it reduces to $a = a''$ for all a. Hence $P_0 = P$ is a Boolean algebra.

Therefore, if $a . \supset . b : \supset : b :: = :: a+b$ is asserted as a property of implication in the propositional calculus P, then the system of logic reduces to the classical. In other

* 'Formal structure I', Theorem 3.

words, in the classical logic 'or' can be expressed purely in terms of implication* and conversely.

THEOREM 7. $(c, ab) = [c, \{c, [(c, a)+(c, b)]\}]$.

This theorem is the result of the application to P_c of a well-known theorem† of product-complements, viz. $(\alpha\beta)' = (\alpha'+\beta')''$, where α and β are arbitrary elements of P. Hence in P_c

$$(ab+c)' = \{(a+c)(b+c)\}' = [(a+c)'+(b+c)']''. \quad (1)$$

But since $(c, ab) = (c, ab+c)$ which $= [c, (a+c)(b+c)]$, (1) easily reduces to

$$(c, ab) = [c, \{c, [(c, a)+(c, b)]\}]$$

which is precisely our theorem.

4. *The correspondence* $x \rightarrow (x, b)$.

THEOREM 8. *The correspondence* $x \rightarrow (x, b)$ *preserves products; in symbols,* $[(xy, b) = (x, b)(y, b)]$.

PROOF. It is sufficient to prove that the proposition $t < (x, b)(y, b)$ is equivalent to the proposition $t < (xy, b)$. Now, the proposition $t < (x, b).(y, b)$ is the same as the simultaneous assertion of the two propositions $t < (x,b)$ and $t < (y, b)$ which by D are respectively equivalent to the propositions $tb < x$ and $tb < y$. But these two propositions are together equivalent to the single proposition $tb < xy$ which by D again is equivalent to $t < (xy, b)$.

THEOREM 9. *The correspondence* $x \rightarrow (x, b)$ *preserves quotients; in symbols,* $[(x, y), b] = [(x, b), (y, b)]$.

For, the right side of the equation to be proved, is by A(iii) the same as $[x, b.(y, b)]$, which by Theorem 2 (iii)

* *cf.* Hilbert and Ackermann, *Grundzuge der Theoritischen Logik*' p. 8, where it is said 'Bemerkenswert ist, daß die Verknupfung v durch→ allein; ohne Anwendung der Negation ausdruckt werden kann. Es gilt namlich XVY aq $(X \rightarrow Y) \rightarrow Y$.'

† For proof, see my paper 'Ideal Theory in Boolean Algebra and its Application to Deductive Systems', *Proc. Ind. Acad. Sc.* 14 (1941) 676, Th. 17.

is the same as (x, by); this by A(iii) again is the same as $[(x, y), b]$ and hence the result follows.

THEOREM 10. *If P is a Boolean algebra, the correspondence $x \to (x, b)$ preserves sums.*

PROOF. P being a Boolean algebra, we have to prove that $(x+y, b) = [(x, b)+(y, b)]$.

Now by Theorem 4 (ii), $(b, a) = a'+b$, we have $(x+y, b) = b'+(x+y)$ which $= (b'+x)+(b'+y) = (x, b)+(y, b)$ from Theorem 4 (ii) again.

I am not able to show that the converse of the theorem holds. It is however easy to prove that if $(x+y, b) = (x, b)+(y, b)$ for all x, y, b, then every normal element is simple (which by no means implies that P is a Boolean algebra). For, $x''+x' = (x, x')+(x', x) = (x, x+x')+(x', x+x') = (x+x', x+x') = 1$ by Theorems 3 and 1 Cor (i) respectively.

On the other hand, it is not known whether the assumption $x'+x'' = 1$ for all x is sufficient to prove that $(x+y, b) = (x, b)+(y, b)$.

5. *Verification for Intuitionistic Logik.*

Lastly, we have to show explicitly, that the interpretation of the implication operation as relative product-complement is valid in Intuitionistic logic. Let us take for instance, Heyting's* formulation of Intuitionistic logic based on the following axioms and rules of inference.

H_1. $a \colon \supset \colon a.\ a.$

H_2. $a.\ b \colon \supset \colon b.\ a.$

H_3. $a.\ \supset .\ b \colon \supset \colon ac.\ \supset .\ bc.$

H_4. $a.\ \supset .\ b.\ b.\ \supset .\ c \colon \supset \colon a.\ \supset .\ c.$

H_5. $b \colon \supset \colon a.\ \supset .\ b.$

H_6. $a.\ a \supset b \colon \supset \colon b.$

H_7. $a.\ \supset .\ a+b.$

H_8. $a+b.\ \supset .\ b+a.$

* A. Heyting: 'Die Formalen Regeln der Intuitionistischen Logik', *Sitzungsberichte der Preuss. Akad. der Wissenschaften*, 45 (1930), pp. 42-56.

H_9. $a \supset c. \ b \supset c: \ \supset: a+b. \supset. \ c.$

H_{10}. $a': \supset: a. \supset. \ b.$

H_{11}. $a. \supset. \ b. \ a. \supset. \ b': \supset: b'. \supset. \ a'.$

Rules of Inference.

1. If a and $a \supset b$ are asserted, then b can be asserted.
2. If a and b are asserted, then $a. b$ can be asserted.

It can easily be shown from the above postulates that the totality of propositions and propositional expressions form a distributive lattice; also, as seen in my previous paper,* the negation a' of any propositional expression a is its product-complement in P.

Now, in the first place, if we interpret the asserted implications in the above postulates H_1-H_{11} as '$<$' in the lattice-order and the other implications as relative product-complements, it is easy to show that H_1-H_{11} follow from the properties of quotients already proved in the previous sections. Thus, H_1, H_2, H_7 and H_8 are simple lattice-properties; H_4, H_5 H_6 and H_9 coincide with our Theorems 2(ii), 2(i), D_1 and 5 respectively. Finally, H_3 follows immediately from D_1 and the lattice-properties; H_{10} is the result of applying Theorem 2 (iv) to the expression $0<b$ while H_{11} is obtained from Theorems 8 and 2 (i).

Conversely, assuming the connection between asserted and considered implication stated in A_1 (and replacing asserted implication by $<$) we can easily derive A_2 and A_3 from H_1-H_{11}; for, A_2 is simply H_6 while A_3 coincides with the Theorem† 2.27 (viz. $a: \supset: b. \supset. \ c.: \supset\subset:. \ ab. \supset. \ c.$) of Heyting's.

Hence it follows by § 1, that Heyting's implication operation is the same as the relative product-complement.

* Formal Structure I, page 52.

† *Ibid*, page 46.

I wish to make the following correction in my paper 'Formal Structure I', p. 61 line 8. *For* 'Axiomensystem fur den Intuitionistic Logic' *read* 'Die Formalen Regeln der Intuitionistischen Logik.'

THE LOWER ORDER OF THE ZEROS OF AN INTEGRAL FUNCTION

BY

S. M. SHAH, *Aligarh*

[Received in revised form, 4 December, 1941]

1. Let $f(z)$ be an integral function of finite order ρ. We write

$$f(z) = z^k e^{Q(z)} P(z),$$

where $Q(z)$ is a polynomial of degree q and $P(z)$ a canonical product of order ρ_1. If $q \leqslant \rho_1$, then we have

$$\rho = \limsup_{r\to\infty} \frac{\log \log M(r)}{\log r} = \rho_1 = \limsup_{r\to\infty} \frac{\log^+ n(r)}{\log r},$$

where $M(r)$ and $n(r)$ have their usual meanings. If $\rho > 0$ is a non-integer, then if $\log \log M(r)/\log r$ tends to a limit, then $\log n(r)/\log r$ tends to the same limit and conversely.

We inquire whether there is any relation between the two inferior limits

$$\lambda = \liminf_{r\to\infty} \frac{\log \log M(r)}{\log r}; \; \lambda_1 = \liminf_{r\to\infty} \frac{\log^+ n(r)}{\log r}. \qquad (1)$$

In this note I prove such a relation in the case of functions of positive order $\rho < 1$. I also prove some results analogous to those proved by J. M. Whittaker* who calls λ the lower order of $f(z)$. λ_1 may similarly be called the lower order of zeros of $f(z)$.

2. THEOREM 1. *If $f(z)$ be of order ρ, $0 < \rho < 1$ then*

$$\lambda_1 \leqslant \lambda \leqslant \lambda_1/(1 + \lambda_1 - \rho). \qquad (2)$$

The first inequality of (2) is an immediate consequence of Jensen's theorem. (2) is a best possible result. (Ex. 1 below).

* J. M. Whittaker, *Jour. Lond. Math. Soc.* 8, (1933), 20—27.

THEOREM 2. *Given* ρ, $0 < \rho < \infty$ *there exists a canonical product* $f(z)$ *of order* ρ *for which* $\lambda > \lambda_1$.

Let $0 < r_1 \leqslant r_2 \ldots$ be the moduli of the zeros of $P(z)$, modulus of a multiple zero being repeated according to its multiplicity.

THEOREM 3.

$$\lambda_1 \leqslant \rho_1 \liminf_{n\to\infty} \frac{\log r_n}{\log r_{n+1}}. \tag{3}$$

Example 3 provides a case where the inequality sign holds.

THEOREM 4. *If* $f(z)$ *is an integral function of irregular growth and positive order* ρ *and if* $\lambda_1 < \mu < \rho$, *then* $f(z)$ *is of the form*

$$z^k e^{g(z)} f_1(z)\, f_2(z),$$

where $f_1(z)$ *is a canonical product of order less than or equal to* μ *and* $f_2(z)$ *is a canonical product of order* ρ *which satisfies the condition*

$$\lambda_1 \geqslant \mu \liminf_{m\to\infty} \frac{\log R_m}{\log R_{m+1}},$$

where R_m *are the moduli of the zeros of* $f_2(z)$ *and* $g(z)$ *is a polynomial of degree less than or equal to* ρ.

3. PROOF OF THEOREM 1. We have*

$$f(z) = Az^k \prod_1^\infty \left(1 - \frac{z}{z_n}\right)$$

$$\log M(r) < A \log r + BI < AI,$$

where

$$I = \int_0^\infty \frac{n(x)}{x} \frac{r}{(x+r)} dx.$$

By hypothesis there is an indefinitely increasing sequence $x_1, x_2, \ldots$ for which $n(x_n) < x_n^{\lambda_1+\varepsilon}$. Also $n(x) < x^{\rho+\varepsilon}$ for $x \geqslant m > 1$. Let $R = x_n$; $\theta = \lambda_1 + 1 - \rho - \varepsilon$; $r = R^\theta$.

* A denotes a positive constant not necessarily the same at each occurrence.

Then

$$I(r)=\int_0^m+\int_m^R+\int_R^\infty<O(1)+n(R)\log R+\int_R^\infty rx^{\rho+\varepsilon-2}dx.$$

$$<O(1)+R^{\lambda_1+\varepsilon}\log R+\frac{rR^{\rho+\varepsilon-1}}{1-(\rho+\varepsilon)}.$$

Hence

$$\liminf_{r\to\infty}\frac{\log\log M(r)}{\log r}\leqslant\frac{\lambda_1}{\lambda_1+1-\rho}. \tag{4}$$

Cor. 1. *If* $0<\rho<1$ *and either* λ *or* $\lambda_1=0$ *then* $\lambda=\lambda_1$. This result is not true if $\rho=1$. (Ex. 2).

Cor. 2. *Given* $\rho=\lambda$ *and* $0<\rho<1$ *then* $\rho=\rho_1=\lambda=\lambda_1$.

This is a well-known result and holds for functions of non-integral order. From (4) and the hypothesis we have

$$\rho=\lambda\leqslant\lambda_1/(\lambda_1+1-\rho).$$

Hence

$$(\rho-\lambda_1)\ (\rho-1)\geqslant 0 \text{ or } \rho\leqslant\lambda_1 \text{ and so } \rho=\lambda_1.$$

4. Proof of Theorem 2. Let

$$a_n=2^{2^{n-1}}\ (n\geqslant 1),\ a_0=0,\ k=1+[\rho],\ \xi_n=[a_n^{\rho/k}]-[a_{n-1}^{\rho/k}];$$

$f(z)=\prod\limits_1^\infty\left(1+\frac{z^k}{a_n}\right)^{\xi_n}$. $f(z)$ is a canonical product. Further

$$n(a_n^{1/k})=k(\xi_1+\cdots+\xi_n)\sim ka_n^{\rho/k}$$

$$n(a_n^{1/k}-1)\sim ka_{n-1}^{\rho/k}=ka_n^{\rho/2k}.$$

Hence $f(z)$ is of order ρ and $\lambda_1=\rho/2$. Further

$$\log M(r)=\sum_1^\infty\xi_n\log\left(1+\frac{r^k}{a_n}\right)>\xi_{n-1}\log\left(1+\frac{R}{a_{n-1}}\right)$$

$$+\xi_n\log\left(1+\frac{R}{a_n}\right),$$

where $R=r^k$. Let $a_{n-1}\leqslant R<a_n$. By splitting up this interval into (1) $a_n>R\geqslant a_n/\log n$ (2) $a_n/\log n>R\geqslant a^{1-\rho/2k}$ (3) $a_n^{1-\rho/2k}>R\geqslant a_{n-1}$, we obtain

$$\lambda\geqslant\frac{\rho}{2-(\rho/k)}.$$

It is easily seen that $\lambda = \rho/\{2-(\rho/k)\}$. $f(z)$ is the required function for which $0<\lambda_1<\lambda<\rho_1=\rho$.

5. PROOF OF THEOREM 3. Let

$$\alpha = \liminf_{r\to\infty} \frac{\log r_n}{\log r_{n+1}}.$$

Then if $\beta = \alpha+\varepsilon$, we have $\log r_n < \beta \log r_{n+1}$ for an infinity of n. For this sequence of values $n=N$ say,

$$\frac{\log N}{\log r_N} > \frac{\log N}{\beta \log r_{N+1}} = \frac{\log N}{\log (N+1)} \cdot \frac{\log (N+1)}{\beta \log r_{N+1}}.$$

Hence

$$\rho_1 \geqslant \limsup_{N\to\infty} \frac{\log N}{\log r_N} \geqslant \frac{1}{\beta} \liminf_{N\to\infty} \frac{\log (N+1)}{\log r_{N+1}} \geqslant \frac{\lambda_1}{\beta}.$$

Since ε is arbitrary, $\lambda_1 \leqslant \alpha\rho_1$.

6. PROOF OF THEOREM 4. Let

$$f(z) = z^k e^{Q(z)} P(z), \text{ where}$$

$$P(z) = \prod_1^\infty \left\{ \left(1-\frac{z}{z_n}\right) \exp\left(\frac{z}{z_n}+\ldots+\frac{1}{p}\left(\frac{z}{z_n}\right)^p\right) \right\} = \prod_1^\infty E\left(\frac{z}{z_n}, p\right).$$

Since $\lambda < \rho$, we have $\rho_1 = \rho$. Consider the points

$$y = y(n) = \frac{\log n}{\log r_n}, \; n = 1, 2, 3, \ldots$$

We know that for a sequenceof values of n, $\log n/\log r_n \to \rho$ and for another sequence, $\log n/\log r_n \to \lambda_1$. Choose an integer $h>0$ such that $\lambda_1+1/h<\mu$. Take the least integer $a_1>1$ such that $y(a_1)\geqslant\mu$ and the least integer $b_1>a_1$ such that

$$y(b_1) < \lambda_1 + \frac{1}{1+h}.$$

Now choose the least integer c_1 such that $a_1<c_1\leqslant b_1$ and

$$y(c_1) \leqslant y(n) \text{ for } a_1 \leqslant n \leqslant b_1.$$

Choose the largest integer d_1 such that $a_1\leqslant d_1<c_1$ and such that $y(d_1)\geqslant\mu$. If there are zeros of $f(z)$ having suffixes greater than d_1 and less than c_1, then we say these

zeros form a set E_1. We choose now the least integer $a_2 > b_1$ such that $y(a_2) \geqslant \mu$. Take the least integer $b_2 > a_2$ such that

$$y(b_2) < \lambda_1 + \frac{1}{h+2}.$$

We choose c_2 and d_2 as before. If there are any zeros having suffixes greater than d_2 and less than c_2, we put these zeros in E_1. We repeat this and note that $y(b_k) < \lambda_1 + 1/(h+k)$, where $k \geqslant 1$. Let

$$\prod_{z_n C E_1} E\left(\frac{z}{z_n}, p\right) = \prod_{z_n C E_1} E\left(\frac{z}{z_n}, \nu\right) \exp\left\{ \sum_{z_n C E_1} \left(\frac{1}{1+\nu} \frac{z^{\nu+1}}{z_n^{\nu+1}} + \dots + \frac{1}{p} \frac{z^p}{z_n^p} \right) \right\}$$

$$= f_1(z)\, e^{g_1(z)},$$

where ν is the genus of $f_1(z)$, and $\nu \leqslant [\mu]$. If $\nu = p$, then $g_1(z) = 0$. If E_1 is an empty set, then $f_1(z) = 1$, $g_1(z) = 0$. Let

$$f_2(z) = \prod_{z_n \notin E_1} E\left(\frac{z}{z_n}, p\right).$$

$f_1(z)$ is a canonical product of order less than or equal to μ, since for $z_n C E_1$, $y(n) \leqslant \mu$.

Further let $R_1, R_2, \dots$ denote the zeros of $f_2(z)$. Then if the zero corresponding to the point d_k be R_m, then that corresponding to the point c_k will be R_{m+1}. Now

$$\frac{\log d_k}{\log r_{d_k}} = \frac{\log d_k}{\log R_m} \geqslant \mu; \quad \frac{\log c_k}{\log r_{c_k}} = \frac{\log c_k}{\log R_{m+1}} < \lambda_1 + \frac{1}{h+k}$$

$$\frac{\log R_m}{\log R_{m+1}} < \frac{1}{\mu} \left(\lambda_1 + \frac{1}{h+k}\right) \frac{\log d_k}{\log c_k} < \frac{\lambda_1}{\mu} + \frac{1}{\mu(h+k)}.$$

Since k tends to infinity with m

$$\liminf_{m\to\infty} \frac{\log R_m}{\log R_{m+1}} \leqslant \frac{\lambda_1}{\mu}.$$

Further

$$f(z) = z^k e^{Q(z)} P(z)$$
$$= z^k e^{Q(z)} f_1 f_2 e^{g_1(z)}$$
$$= z^k e^{g(z)} f_1(z) f_2(z),$$

where $g(z) = Q(z) + g_1(z)$ is a polynomial of degree less than or equal to ρ.

7. Ex. 1. *Given ρ and λ_1, $0 < \rho < 1$, $0 < \lambda_1 < \rho$ there exists an integral function $f(z)$ of order ρ for which $\lambda = \lambda_1/(1+\lambda_1-\rho)$.*

Proof. Let $a_1 = 2$; $a_n = a_{n-1}^{\rho/\lambda_1}$; $\xi_n = [a_n^\rho] - [a_{n-1}^\rho]$,

$$f(z) = \prod_1^\infty \left(1 + \frac{z}{a_n}\right)^{\xi_n}.$$

Ex 2. Let $a_1 = 2$; $b_m = a_m^{m+1}$; $a_{m+1} = 2^{b_m}$

$$f(z) = \frac{\prod_2^\infty \left\{1 + \dfrac{z}{n \log^2 n}\right\} \prod_{r=1}^\infty \left\{1 + \dfrac{z}{b_r \log^2 b_r}\right\}^{b_r - a_r + 1}}{\prod_{r=1}^\infty \left\{\prod_{n=a_r}^{n=b_r} \left(1 + \dfrac{z}{n \log^2 n}\right)\right\}}.$$

$f(z)$ is a canonical product of order one and genus zero. Here $\lambda = 1$ and $\lambda_1 = 0$.

Ex 3. Let $a_1 = 2$; $b_m = 2^{a_m}$; $a_{m+1} = 2^{b_m}$.

$$f(z) = \prod_{m=1}^\infty \left\{ \prod_{b_{m-1} < \sqrt{n} < a_m} \left(1 - \frac{z}{\sqrt{n}}\right) \exp\left(\frac{z}{\sqrt{n}} + \frac{z^2}{2n}\right) \right.$$
$$\left. \times \prod_{a_m \leqslant n \leqslant b_m} \left(1 - \frac{z}{n}\right) \exp\left(\frac{z}{n} + \frac{z^2}{2n^2}\right) \right\}. \quad (5)$$

$f(z)$ is a canonical product of order 2. Here $\rho_1 = \rho = 2$; $\lambda_1 = 1$. Further

$$\frac{\log r_m}{\log r_{m+1}} \to 1 \text{ as } m \to \infty. \quad (6)$$

Postscript. Theorem 4 also holds if instead of assuming that $f(z)$ is of irregular growth, we assume that $\lambda_1 < \rho_1$. If $\lambda_1 < \mu < \rho_1$, then $f(z)$ is of the form $z^k f_1 f_2 e^{g(z)}$ where f_2 is a canonical product of order ρ_1 satisfying the condition stated in Theorem 4, while f_1 and g have the same meanings as before.

A PROPERTY OF THE MAXIMUM MODULUS OF INTEGRAL FUNCTIONS*

BY

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1. Let E be a measurable set in $(0, \infty)$. Let $E(r)$ be the measure of the portion of E in $(0, r)$. The quantities

$$\overline{\lim_{r\to\infty}} \frac{E(r)}{r} \text{ and } \underline{\lim_{r\to\infty}} \frac{E(r)}{r}$$

are respectively called *the upper* and *the lower density* of E in $(0, \infty)$. If the two are equal, the common value is called *the density* of E. It is evident that

$$0 \leqslant \underline{\lim_{r\to\infty}} \frac{E(r)}{r} \leqslant \overline{\lim_{r\to\infty}} \frac{E(r)}{r} \leqslant 1.$$

2. Let $f(z)$ be an integral function and $M(r,f) = \max\limits_{|z|\leqslant r} |f(z)|$. The order ρ of the function $f(z)$ is defined by the relation

$$\rho = \overline{\lim_{r\to\infty}} \frac{\log\log M(r,f)}{\log r}. \tag{1}$$

It is obvious that, except in the case when $f(z)$ is a constant of modulus not exceeding unity, the number ρ defined by (1) satisfies the relation $0 \leqslant \rho \leqslant \infty$. In this paper we suppose that $\rho \geqslant 0$ is finite. If $f(z)$ is a function of finite order ρ such that

$$\lim_{r\to\infty} \frac{\log\log M(r,f)}{\log r} \neq \rho, \tag{2}$$

* An abstract of this paper was communicated to the twelfth conference of the Indian Mathematical Society held at Aligarh during December 1941.

then the function is said to be of *regular growth*. If $\rho = 0$, (2) is obviously true so that in this case the function is always of regular growth. But in this case we shall require a finer distinction. If $f(z)$ is of order zero, we define its logarithmic order l by the relation

$$l = \overline{\lim_{r \to \infty}} \frac{\log \log M(r, f)}{\log \log r}. \tag{3}$$

It is a well-known fact that $l \geqslant 1$ unless $f(z)$ is a constant. We suppose throughout that l is finite. So the theorems of this paper will apply to functions of finite order and, if the order is zero, to functions of finite logarithmic order. Here also, if the maximum limit in (3) can be replaced by a regular limit for a function, such a function is said to be of *regular logarithmic growth*.

2.1. The relation (1) implies that there is a sequence $\{r_n\}$, $(0 \leq r_1 < r_2 \ldots < r_n \to \infty)$ such that

$$\lim_{n \to \infty} \frac{\log \log M(r_n, f)}{\log r_n} = \rho. \tag{4}$$

We shall say that a set E in $(0, \infty)$ is a *set of regularity for the function $f(z)$ if*

$$\lim \frac{\log \log M(r, f)}{\log r} = \rho, \tag{5}$$

as $r \to \infty$ over E. The relation (4) implies that such sets exist. The main object of th s paper is to study the properties of sets of regularity for functions of finite order, and when the order is zero for functions of finite logarithmic order, the sets of regularity in the latter case being defined similarly.

2.2. If a function is of regular growth or of regular logarithmic growth, the interval $(0, \infty)$ itself is a set of regula ity and all other sets of regularity are included in it. The first question that arises is whether in all cases there exists a maximum p ssible set of regularity in the sense that if E is such a set, then (i) E is a set of regularity and (ii) if E_1 is any set of regularity then all points of E_1 after a certain stage are contained in E. I shall prove pre-

sently that, except in the case of functions of regular growth, such sets do not exist. So it appears that, in general, it is not possible to specify the set of *all points* E such that (5) is true as $r \to \infty$ over E. I shall however prove that every function of finite order possesses at least one set of regularity which is measurable and has upper density one. I prove also that if a function has *a set of regularity of positiv lower density, then it is of regular growth.* I discuss some allied questions and prove by an example that the above results are the best possible.

2 3. I shall now prove that, *except in the case of functions of regular growth, it is not possible to find a maximum possible set of regularity as explained in* § 2.2. Suppose $f(z)$ is of order ρ but not of regular growth. If

$$\lambda = \varliminf_{r\to\infty} \frac{\log\log M(r, f)}{\log r},$$

then $\lambda < \rho$. Then there 's a sequence $\{r_n\}$ for which (4) holds and another sequence $\{t_n\}$, $(0 \leqslant t_1 < t_2 \ldots < t_n \to \infty)$ for which

$$\lim_{n\to\infty} \frac{\log\log M(t_n, f)}{\log t_n} = \lambda. \tag{6}$$

Let E be a set of regularity of $f(z)$. Let μ be such that $\lambda < \mu < \rho$. Using (4), (6) and the fact that $\dfrac{\log\log M(r, f)}{\log r}$ is continuous for all large r, we can find a sequence $\{r_n(\mu)\}$, $[0 < r_1(\mu) < r_2(\mu) \ldots < r_n(\mu) \to \infty]$ such that

$$\frac{\log\log M\{r_n(\mu), f\}}{\log r_n(\mu)} = \mu \tag{7}$$

while *no* $r_n(\mu)$ *be'ongs to* E. If p is a sufficiently large integer, all the sequences corresponding to $\mu = \rho - 1/p$ are defined. Hence we can find a sequence of integers $\{p_n\}$, $(p_1 < p_2 < p_3 \ldots < p_n \to \infty)$ such that the sequence $\{r_n(\rho - 1/p_n)\}$ tends to infinity. Th's sequence is evidently a set of regularity by (7), while no point of this set

belongs to E. This proves the statement made at the beginning of this article.

2.4. In spite of the result in the previous paragraph we can get an idea of all possible sets of regularity as follows. Let $S(\varepsilon)$ be the set of all points in $(0, \infty)$ such that

$$\rho-\varepsilon \leqslant \frac{\log \log M(r, f)}{\log r} \leqslant \rho+\varepsilon.$$

Consider the collection of sets $C=\{S(1), S(\frac{1}{2}), \ldots\}$. A sequence $\{r_n\}$ tending to infinity such that r_n belongs to $S(1/p_n)$ where $p_n \to \infty$ is a set of regularity and conversely. But it is not possible in general to specify the collection of all such sequences as a point set which is also a set of regularity.

2.5. Similar statements hold for functions of finite logarithmic order.

3. We now prove the main results of this paper.

THEOREM. 1. *Let $f(z)$ be a function of finite order $\rho \geqslant 0$. Then there exists a set of regularity $R(f)$ of $f(z)$ whose upper density is one.*

3.1. The proof of the above theorem is based on the following

LEMMA 1. *Let $f(z)$ be of order ρ and $0<\delta<\rho$. Let $E(\delta)$ denote the set of all points in $(0, \infty)$ where*

$$\frac{\log \log M(r, f)}{\log r} \leqslant \rho-\delta.$$

Let V_0 denote the interval $0 \leqslant r \leqslant 1$ and V_n the interval $\exp\{(n-1) \log (n-1)\} \leqslant r \leqslant \exp\{n \log n\}$ for $n \geqslant 1$. Then there are an infinity of intervals among $\{V_n\}$ containing no point of $E(\delta)$.

PROOF. If the lemma is not true, then there is an n_0 such that for all $n \geqslant n_0$, V_n contains a point of $E(\delta)$. Let r_n be such a point in V_n. Let r be given. Then it is

possible to find a p such that r is in V_p. Hence, if r is large enough,

$$\log\log M(r,f) \leqslant \log\log M(r_{p+1},f) \leqslant (\rho-\delta)\log r_{p+1}. \quad (8)$$

But we have

$$\begin{aligned} r_{p+1} &\leqslant \exp\left\{(p+1)\log(p+1)\right\} \\ &\leqslant r\exp\left\{(p+1)\log(p+1)-(p-1)\log(p-1)\right\} \\ &\leqslant r\exp\left[2\log p\left\{1+o(1).\right\}\right]. \end{aligned} \quad (9)$$

Also

$$\log r = p\log p\left\{1+o(1)\right\}. \quad (10)$$

Using (9) and (10) in (8) we get

$$\log\log M(r,f) \leqslant (\rho-\delta)\log r\left\{1+o(1)\right\},$$

which shows that $f(z)$ is of order $\leqslant \rho-\delta$ which contradicts the hypothesis that $f(z)$ is of order ρ. So the lemma is true.

3.2. PROOF OF THEOREM 1. Taking $\delta = 1, 1/2, \ldots, 1/n, \ldots$ successively in Lemma 1, we see that there exists a sequence of integers $\{p_n\}$, $(p_1 < p_2 \ldots < p_n \to \delta)$ such that in the interval V_{p_n} we have

$$\frac{\log\log M(r,f)}{\log r} \geqslant \rho - \frac{1}{n}. \quad (11)$$

Let $R(f)$ be the set of intervals $\{V_{p_n}\}$. Then (11) shows that $R(f)$ is a set of regularity. Also it is measurable. Moreover its upper density is not less than

$$\lim_{n\to\infty} e^{-p_n\log p_n}\left\{e^{p_n\log p_n} - e^{(p_n-1)\log(p_n-1)}\right\}$$

which is equal to one. So the theorem is proved.

3.3. An exactly similar method proves* the following

* Even if $\rho = \infty$ in Theorem 1, the result remains true. Only, in this case, $\log r$ is not a proper comparison function for $\log\log M(r,f)$ and so the theorem possesses no interest. A similar statement holds for functions of infinite logarithmic order.

THEOREM 2. *If $f(z)$ is a function of order zero but of finite logarithmic order, there exists a set of regularity $R(f)$ having upper density one.*

4. Next we prove the following

THEOREM 3. *Let $f(z)$ be a function of order ρ. If it has a set of regularity $R(f)$ having positive lower density, then $f(z)$ is of regular growth.*

PROOF : Let $I_n(\lambda)$ denote the interval $\lambda^{n-1} \leqslant r \leqslant \lambda^n$ and $I_0(\lambda) = (0, 1)$ where λ is any number greater than one. Since $R(f)$ has positive lower density, it follows that there is a $\delta > 0$ and an integer n_0 such that the measure of $R(f)$ in $(0, \lambda^n)$ is $\geqslant \delta \lambda^n$, for $n \geqslant n_0$. Hence the measure of $R(f)$ in $I_n(\lambda)$ is $\geqslant \lambda^{n-1}(\lambda\delta - 1)$. Let λ be any number greater than $1/\delta$. Then any $I_n(\lambda)$, $(n \geqslant n_0)$ contains a point of $R(f)$. Let r_n be a point of $R(f)$ in $I_n(\lambda)$. Let r be any large number. We can find an integer p such that r is in $I_p(\lambda)$. Hence

$$\log \log M(r, f) \geqslant \log \log M(r_{p-1}, f) = \{\rho + o(1)\} \log r_{p-1}.$$

But

$$r_{p-1} \geqslant \lambda^{p-2} \geqslant \frac{1}{\lambda^2} \cdot \lambda^p \geqslant \frac{r}{\lambda^2}.$$

Hence, as $r \to \infty$,

$$\log \log M(r, f) \geqslant \{\rho + o(1)\}(\log r - 2 \log \lambda)$$

so that

$$\varliminf_{r \to \infty} \frac{\log \log M(r, f)}{\log r} \geqslant \rho.$$

This relation along with (1) proves the theorem.

4.1. We shall say that a function is of *totally irregular growth* if *every* set of regularity of the function has *lower density zero*. Theorem 3 at once leads to

THEOREM 4. *Every function of finite order is either of regular growth or of totally irregular growth.*

4 2. Analogues of Theorems 3 and 4 are true for functions of finite logarithmic order. Theorem 4 leads us to expect that there exist functions of totally irregular growth. We shall now construct an example to show that this is true so that the result of Theorem 1 is the best possible. Let $e_0 = 1$, $e_p = \exp(e_{p-1})$, $p \geqq 1$. Let $0 < \sigma < 1$. Let $H(z)$ be the canonical product which has $[e_{2p}^{\sigma}]$ zeros in the interval $(e_{2p}, e_{2p}+1)$. Then it is easy to see that $\log M(r, H)$ is given by the expression

$$\log M(r, H) = \sum_{n=1}^{\infty} [e_{2n}^{\sigma}] \log \left(1 + \frac{r}{e_{2n}} \right) + O(1), \qquad (12)$$

as $r \to \infty$. Using the well known properties of the exponential and logarithmic functions and remembering the definition of the sequence $\{e_n\}$ and that $0 < \sigma < 1$, a little calculation gives the following results:—

(i) If $e_{2p} \leqslant r \leqslant e_{2p+2}$, then as $r \to \infty$,

$$\log M(r, H) = [e_{2p}^{\sigma}] \log \left(1 + \frac{r}{e_{2p}} \right) + e_{2p+2}^{\sigma} \log \left(1 + \frac{r}{e_{2p+2}} \right) + [e_{2p-2}^{\sigma}] \log r \left\{ 1 + o(1) \right\}. \qquad (13)$$

(ii) If $e_{2p} \leqslant r \leqslant e_{2p+1}$, (13) gives*

$$\log M(r, H) = [e_{2p}^{\sigma}] \log \left(1 + \frac{r}{e_{2p}} \right) + O\left\{ (\log r)(l_2 r)^{\sigma} \right\}. \qquad (14)$$

(iii) If $e_{2p+1} \leqslant r \leqslant e_{2p+2}$, (13) gives

$$\log M(r, H) = e_{2p+2}^{\sigma} \log \left(1 + \frac{r}{e_{2p+2}} \right) + O\left\{ (\log r)^{1+\sigma} \right\}. \qquad (15)$$

The result (14) shows that $H(z)$ is of order σ, which might also have been derived from a well known property of canonical products. Also, if

$$\frac{e_{2p+1}}{p} \leqslant r \leqslant e_{2p+1}, \qquad (16)$$

we get from (14)

$$\log M(r, H) = (\log r)^{1+\sigma} \left\{ 1 + o(1) \right\},$$

* $l_k r = \log \log \ldots \log (r)$, the logarithm being repeated k times.

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so that the intervals (16) form a set E of upper density one such that

$$\lim_{r\to\infty} \frac{\log\log M(r, H)}{\log\log r} = 1+\sigma<\infty,$$

as $r\to\infty$ over E. *This result is much more than what is necessary to prove that $H(z)$ is of totally irregular growth.* It can further be shown that the lower order of growth of $\log M(r, H)$ is less than than that of $\{\log r\}^{1+\varepsilon}$ however small $\varepsilon>0$ may be. With the same notation we construct the canonical product $G(z)$ which has $[e^2_{2p-1}]$ zeros in the interval $(e_{2p}, e_{2p}+1)$. The function $G(z)$ will serve to show that Theorem 2 is a best possible result.

5. Let $g(z)$ and $h(z)$ be two integral functions and let

$$f(z)=g\{h(z)\}. \tag{17}$$

The function $f(z)$ is an integral function and is called a composite integral function. A theorem due to Polya* states that *if $f(z)$ is of finite order, then either $h(z)$ is a polynomial or else $g(z)$ is a function of order zero.* If $h(z)$ is a polynomial of degree p and $g(z)$ is of order γ, the order of $f(z)$ is $p\gamma$. If $h(z)$ is not a polynomial, then $g(z)$ is of order zero. Suppose the order of $h(z)$ is σ and $g(z)$ has finite logarithmic order l. Let ρ be the order of $f(z)$. Then from the obvious inequality

$$M(r,f)\leqslant M\{M(r, h), g\} \tag{18}$$

we get the relat on

$$\rho\leqslant l\sigma. \tag{19}$$

The question now arises under what conditions we can assert $\rho=l\sigma$. The precise answer is given by the following

THEOREM 5. *Let $f(z)$, $g(z)$, $h(z)$ be defined by the relation (17). Let $f(z)$ be of order ρ, $g(z)$ of logarithmic order l and $h(z)$ of order σ. Then*

$$\rho=l\sigma, \tag{20}$$

* *Jour. Lond. Math. Soc.*, 1 (1926), 12-15.

unless both the functions $h(z)$ and $g(z)$ are of totally irregular growth.

The proof of this theorem follows easily from Theorem 4 and the following result due to Polya*:

LEMMA 2. *Let $f(z), g(z)$ and $h(z)$ be related as in (17). Let $h(0)=0$. Then there is a definite number c in $0 < c < 1$ independent of f, g and h such that*

$$M(2r, f) \geqslant M\left\{cM(r, h), g\right\}.$$

5.1. In the exceptional case mentioned in Theorem 5, it can happen that $\rho < l\sigma$. Consider the function $H(z)$ constructed in §4.2. Let

$$\mu_p = M(e_p, H).$$

Now let $g(z)$ be the canonical product having $[(\log \mu_{2p+1})^2]$ zeros in the interval $(\mu_{2p+1}, \mu_{2p+1}+1)$. Then $g(z)$ is of logarithmic order 2, and if Theorem 5 were universally true the order of $f(z)=g\{H(z)\}$ must be 2σ. But a detailed calculation by using (13), (14), (15) and similar results for $g(z)$ shows that the actual order of $g\{H(z)\}$ is zero. This shows that Theorem 5 is a best possible result.

6. Given a function $f(z)$ of positive order ρ, it is possible to find† a positive continuous function $\rho(r)$ having the following properties:

(i) $\rho(r)$ is differentiable to the right and to the left; (ii) $\rho(r) \to \rho$ as $r \to \infty$; (iii) $r \log r\rho'(r) \to 0$ as $r \to \infty$; (iv) $\overline{\lim_{r\to\infty}} \dfrac{\log M(r,f)}{r^{\rho(r)}} = 1$.

Such a function $\rho(r)$ is called a Lindelof's proximate order. Problems already discussed in connection with the equation (1) can now be considered with reference to

* *l.c.*, pp. 12-13.

† G. Valiron, *Lectures on Integral Functions*, pp. 64-67.

the relation (iv) above. The function $f(z)$ is said to be *of perfectly regular growth* if

$$\lim_{r\to\infty} \frac{\log M(r, f)}{r^{\rho(r)}} = 1. \tag{21}$$

If E be a point set in $(0, \infty)$ such that

$$\lim \frac{\log M(r, f)}{r^{\rho(r)}} = 1 \tag{22}$$

as $r \to \infty$ *over* E, we may call E *a set of perfect regularity.* The remarks of §2.3 apply in this case also with reference to the specification of a maximum possible set of perfect regularity. I prove below two theorems which are the analogues of Theorems 1 and 3, but the results are in no way as precise as in these theorems. It appears improbable that such precise theorems could be proved in this case, though I have not succeeded in constructing an example to prove or disprove this assertion.

6.1. *Theorem 6. Let* $\rho(r)$ *be a Lindelof's proximate order of* $f(z)$ *which is of order* ρ. *Let* $E(\delta)$, $(0 < \delta < 1)$, *be the set of points in* $(0, \infty)$ *where*

$$\log M(r, f) \geqslant (1-\delta)\, r^{\rho(r)}.$$

Then the upper density of $E(\delta)$ *is* $\geqslant 1-(1-\delta)^{1/2\rho}$ *or what is the same thing, the lower density of the set* $CE(\delta)$ *complementary to* $E(\delta)$ *is* $\leqslant (1-\delta)^{1/2\rho}$.

Proof. We give the proof for the case $\rho(r) \equiv \rho$. The general case will differ from this only slightly, involving the use of the properties (ii) and (iii) of $\rho(r)$. Suppose that the lower density of $CE(\delta)$ is *greater than* $(1-\delta)^{1/2\rho}$. Then it is possible to find a number $\eta > (1-\delta)^{1/2\rho}$ and less than 1 and an r_0 such that the measure of $CE(\delta)$ in $(0, r)$ is not less than ηr for all $r \geqslant r_0$. Let λ be such that

$$1/\eta < \lambda < (1-\delta)^{-1/2\rho}, \tag{23}$$

so that $\lambda > 1$. Let $\{I_n(\lambda)\}$ be the intervals considered in §4. We can find an n_0 such that if $n \geqslant n_0$ the measure

of $CE(\delta)$ in $I_n(\lambda)$ is not less than $(\eta\lambda^n - \lambda^{n-1}) = \lambda^{n-1}(\eta\lambda - 1) > 0$ so that every $I_n(\lambda)$, $n \geqslant n_0$, contains a point r_n of $CE(\delta)$. If r is any large number, we can find p so that r is in $I_p(\lambda)$. Therefore we have

$$\log M(r, f) \leqslant \log M(r_{p+1}, f) < (1 - \delta)\, r_{p+1}^\rho$$

since r_{p+1} belongs to $CE(\delta)$. Hence

$$\log M(r, f) \leqslant (1-\delta)\,(\lambda^2 r)^\rho$$

so that

$$\varliminf_{r \to \infty} \frac{\log M(r, f)}{r^\rho} \leqslant \lambda^{2\rho}\,(1-\delta) < 1,$$

by (23), which contradicts the hypothesis that

$$\varliminf_{r \to \infty} \frac{\log M(r, f)}{r^\rho} = 1.$$

Hence the lower density of $CE(\delta)$ is $\leqslant (1-\delta)^{1/2\rho}$ and so the theorem is proved.

6.2. THEOREM 7. *Let $f(z)$ and $\rho(r)$ be as in Theorem 6. Let $f(z)$ possess a set E of perfect regularity of lower density $\delta > 0$. Then*

$$\varliminf_{r \to \infty} \frac{\log M(r, f)}{r^{\rho(r)}} \geqslant \delta^{2\rho}.$$

PROOF. As before we take $\rho(r) \equiv \rho$. Let λ be any number greater than $1/\delta$. Then we can find an n_0 such that if $n \geqslant n_0$, the measure of E in $I_n(\lambda)$ is positive. Hence there is a point r_n in $I_n(\lambda)$ for $n \geqslant n_0$. Let r be large enough and let it belong to $I_p(\lambda)$. Then

$$\log M(r, f) \geqslant \{1 + o(1)\}\, r_{p-1}^\rho \geqslant \{1 + o(1)\} \left(\frac{r}{\lambda^2}\right)^\rho.$$

Hence

$$\varliminf_{r \to \infty} \frac{\log M(r, f)}{r^\rho} \geqslant \frac{1}{\lambda^{2\rho}}.$$

Since λ can be taken as near to $1/\delta$ as we like, the theorem follows.

6.3. Theorem 7 leads at once to the following result on taking $\delta = 1$.

THEOREM 8. *A necessary and sufficient condition that $f(z)$ be of perfectly regular growth is that it possesses a set of perfect regularity having density one.*

6.4. Finally it may be noted that although the properties have been proved for the maximum modulus of integral functions, the main results hold for any increasing continuous function of r in $(0, \infty)$, with the same definitions of order as have been utilised for the maximum modulus function.

ON AN EXTENSION OF A THEOREM OF WATSON IN GENERALISED HYPERGEOMETRIC SERIES.

BY

S. C. MITRA, *Dacca.*

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Watson has proved that*

$$ {}_3F_2\left[\begin{matrix} a,\ b,\ c \\ \tfrac{1}{2}(a+b+1),\ 2c \end{matrix};\ 1\right] $$

$$ =\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}+c)\Gamma(\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}b)\Gamma(\frac{1}{2}-\frac{1}{2}a-\frac{1}{2}b+c)}{\Gamma(\frac{1}{2}+\frac{1}{2}a)\Gamma(\frac{1}{2}+\frac{1}{2}b)\Gamma(\frac{1}{2}-\frac{1}{2}a+c)\Gamma(\frac{1}{2}-\frac{1}{2}b+c)}. $$

The object of the present note is to obtain the sum of

$$ {}_3F_2[\alpha_1,\ \alpha_2,\ \alpha_3;\ 2\alpha_3+2m,\ \tfrac{1}{2}(\alpha_1+\alpha_2+1);\ 1], $$

$$ R(4m+1+2\alpha_3-\alpha_1-\alpha_2)>0. $$

The author has proved elsewhere† that

$$ {}_pf_q(\alpha_1,\alpha_2,\ldots,\alpha_p;\ \rho_1+m,\ \rho_2,\ldots,\rho_q;\ x) $$

$$ =\frac{2}{\Gamma(-m)}\sum_{r=0}^{\infty}\frac{\Gamma(\rho_1-1+r)\Gamma(\frac{1}{2}\rho_1+\frac{1}{2}+r)\Gamma(r-m)}{\Gamma(r+1)\Gamma(\frac{1}{2}\rho_1-\frac{1}{2}+r)\,\Gamma(\rho_1+m+r)} $$

$$ \times x^r\,{}_pf_q\left(\begin{matrix}\alpha_1+r,\ \alpha_2+r,\ldots,\alpha_p+r \\ \rho_1+2r,\ \rho_2+r,\ldots,\rho_q+r\end{matrix};\ x\right) $$

where

$$ {}_pf_q(\alpha_1,\alpha_2,\ldots,\alpha_p;\ \rho_1,\rho_2,\ldots,\rho_q;\ x) $$

$$ =\frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\ldots\Gamma(\alpha_p)}{\Gamma(\rho_1)\Gamma(\rho_2)\ldots\Gamma(\rho_q)}\,{}_pF_q(\alpha_1,\alpha_2,\ldots,\alpha_p;\ \rho_1,\rho_2,\ldots,\rho_q;\ x). $$

Let us write $2m$ for m and take m to be a positive integer. Let $\rho_1=2\alpha_3$ and $\rho_2=\frac{1}{2}(\alpha_1+\alpha_2+1)$.

**Proc. Lond. Math. Soc.* (2) 23 (1925), xiii—xv.

†*Proc. Edn. Math. Soc.* (2) 6 (1938), 19-23.

Then

$$\frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)}{\Gamma(2\alpha_3+2m)\Gamma\left[\frac{1}{2}(\alpha_1+\alpha_2+1)\right]} \times {}_3F_2[\alpha_1,\ \alpha_2,\ \alpha_3;\ 2\alpha_3+2m;\ \tfrac{1}{2}(\alpha_1+\alpha_2+1);\ 1]$$

$$= \frac{2}{\Gamma(-2m)}\sum_{r=0}^{2m}\Bigg[\frac{\Gamma(2\alpha_3-1+r)\Gamma(\alpha_3+\frac{1}{2}+r)\Gamma(r-2m)\Gamma(\alpha_1+r)\Gamma(\alpha_2+r)\Gamma(\alpha_3+r)}{\Gamma(r+1)\Gamma(\alpha_3-\frac{1}{2}+r)\Gamma(2\alpha_3+2m+r)\Gamma(2\alpha_3+2r)\Gamma\left[\frac{1}{2}(\alpha_1+\alpha_2+1)+r\right]}\Bigg]$$

$$\times {}_3F_2[\alpha_1+r,\ \alpha_2+r,\ \alpha_3+r;\ 2\alpha_3+2r,\ \tfrac{1}{2}(\alpha_1+\alpha_2+1)+r;\ 1]:$$

By Watson's Theorem, the right side is equal to

$$= \frac{2\Gamma(\frac{1}{2})}{\Gamma(-2m)}\sum_{r=0}^{2m}\Bigg[\frac{\Gamma(2\alpha_3-1+r)\Gamma(\alpha_3+\frac{1}{2}+r)\Gamma(r-2m)\Gamma(\alpha_1+r)\Gamma(\alpha_2+r)}{\Gamma(r+1)\Gamma(\alpha_3-\frac{1}{2}+r)\Gamma(2\alpha_3+2m+r)\Gamma(2\alpha_3+2r)} \times$$

$$\frac{\Gamma(\alpha_3+r)\Gamma(\frac{1}{2}+\alpha_3+r)\Gamma\left[\frac{1}{2}(1-\alpha_1-\alpha_2)+\alpha_3\right]}{\Gamma(\frac{1}{2}+\frac{1}{2}\alpha_1+\frac{1}{2}r)\Gamma(\frac{1}{2}+\frac{1}{2}\alpha_2+\frac{1}{2}r)\Gamma(\frac{1}{2}-\frac{1}{2}\alpha_1+\alpha_3+\frac{1}{2}r)\Gamma(\frac{1}{2}-\frac{1}{2}\alpha_2+\alpha_3+\frac{1}{2}r)}\Bigg].$$

After a little simplification this reduces to

$$\frac{2^{\alpha_1+\alpha_2-2\alpha_3}\Gamma\left[\frac{1}{2}(1-\alpha_1-\alpha_2)+\alpha_3\right]}{\Gamma(-2m)}\sum_{r=0}^{2m}\Bigg[\frac{\Gamma(r-2m)\Gamma(\frac{1}{2}+\alpha_3+r)\Gamma(2\alpha_3-1+r)}{\Gamma(r+1)\Gamma(2\alpha_3+2m+r)\Gamma(\frac{1}{2}-\frac{1}{2}\alpha_1+\frac{1}{2}r+\alpha_3)}$$

$$\times\frac{\Gamma(\frac{1}{2}\alpha_1+\frac{1}{2}r)\Gamma(\frac{1}{2}\alpha_2+\frac{1}{2}r)}{\Gamma(\frac{1}{2}-\frac{1}{2}\alpha_2+\frac{1}{2}r+\alpha_3)\Gamma(\alpha_3-\frac{1}{2}+r)}\Bigg]$$

$$= \frac{2^{\alpha_1+\alpha_2-2\alpha_3}\Gamma\left[\frac{1}{2}(1-\alpha_1-\alpha_2)+\alpha_3\right]}{\Gamma(-2m)}\Bigg[$$

$$\sum_{r=0}^{m}\frac{\Gamma(2r-2m)\Gamma(\frac{1}{2}+\alpha_3+2r)\Gamma(2\alpha_3-1+2r)}{\Gamma(2r+1)\Gamma(2\alpha_3+2m+2r)\Gamma(\frac{1}{2}-\frac{1}{2}\alpha_1+\alpha_3+r)}$$

$$\times\frac{\Gamma(\frac{1}{2}\alpha_1+r)\Gamma(\frac{1}{2}\alpha_2+r)}{\Gamma(\frac{1}{2}-\frac{1}{2}\alpha_2+\alpha_3+r)\Gamma(\alpha_3-\frac{1}{2}+2r)}$$

$$+\sum_{r=0}^{m-1} \frac{\Gamma(2r+1-2m)\Gamma(\alpha_3+\frac{3}{2}+2r)\Gamma(2\alpha_3+2r)}{\Gamma(2r+2)\Gamma(2\alpha_3+2m+1+2r)\Gamma(1-\frac{1}{2}\alpha_1+\alpha_3+r)} \cdot$$

$$\left. \times \frac{\Gamma(\frac{1}{2}+\frac{1}{2}\alpha_1+r)\Gamma(\frac{1}{2}+\frac{1}{2}\alpha_2+r)}{\Gamma(1-\frac{1}{2}\alpha_2+\alpha_3+r)\Gamma(\frac{1}{2}+\alpha_3+2r)} \right]$$

$$= \frac{2^{\alpha_1+\alpha_2-2\alpha_3}\Gamma[\frac{1}{2}(1-\alpha_1-\alpha_2)+\alpha_3]\Gamma(\alpha_3+\frac{1}{2})\Gamma(2\alpha_3-1)\Gamma(\frac{1}{2}\alpha_1)\Gamma(\frac{1}{2}\alpha_2)}{\Gamma(\alpha_3-\frac{1}{2})\Gamma(2\alpha_3+2m)\Gamma(\frac{1}{2}-\frac{1}{2}\alpha_1+\alpha_3)\Gamma(\frac{1}{2}-\frac{1}{2}\alpha_2+\alpha_3)}$$

$$\times {}_7F_6\left[\begin{matrix}\alpha_3-\frac{1}{2},\ \frac{1}{2}\alpha_3+\frac{3}{4},\ \alpha_3,\ \frac{1}{2}\alpha_1,\ \frac{1}{2}\alpha_2,\ -m,\ -m+\frac{1}{2} \\ \frac{1}{2}\alpha_3-\frac{1}{4},\ \frac{1}{2},\ \frac{1}{2}-\frac{1}{2}\alpha_1+\alpha_3,\ \frac{1}{2}-\frac{1}{2}\alpha_2+\alpha_3,\ \alpha_3+m+\frac{1}{2},\ \alpha_3+m\end{matrix};1\right]$$

$$-2m\frac{\Gamma(\alpha_3+\frac{3}{2})\Gamma(2\alpha_3)\Gamma(\frac{1}{2}+\frac{1}{2}\alpha_1)\Gamma(\frac{1}{2}+\frac{1}{2}\alpha_2)}{\Gamma(\alpha_3+\frac{1}{2})\Gamma(2\alpha_3+2m+1)\Gamma(1-\frac{1}{2}\alpha_1+\alpha_3)\Gamma(1-\frac{1}{2}\alpha_2+\alpha_3)}$$

$$\times {}_7F_6\left[\begin{matrix}\alpha_3+\frac{1}{2},\ \frac{1}{2}\alpha_3+\frac{5}{4},\ \alpha_3,\ \frac{1}{2}+\frac{1}{2}\alpha_1,\ \frac{1}{2}+\frac{1}{2}\alpha_2,\ -m+\frac{1}{2},\ -m+1 \\ \frac{1}{2}\alpha_3+\frac{1}{4},\ \frac{3}{2},\ 1-\frac{1}{2}\alpha_1+\alpha_3,\ 1-\frac{1}{2}\alpha_2+\alpha_3,\ \alpha_3+m+1,\ \alpha_3+m+\frac{1}{2}\end{matrix};1\right].$$

It will be noticed that the two Hypergeometric series ${}_7F_6$ are "well-poised" and terminating, and can be summed up by Dougall's theorem*, provided $\alpha_3+2m=\frac{1}{2}(\alpha_1+\alpha_2+1)$.

If m is not a positive integer, the theorem is still true, provided $R(4m+1+2\alpha_3-\alpha_1-\alpha_2)>0$. In this case the hypergeometric series ${}_7F_6$ are non-terminating unless $m-\frac{1}{2}$ is an integer.

* *Proc. Edin. Math. Soc.* 25 (1907), 114—32.

ON A CERTAIN TRANSFORMATION ABOUT GENERALISED HYPERGEOMETRIC SERIES

BY

S. C. MITRA, *Dacca*.

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I have proved elsewhere* that

$$
{}_pf_q\,(\alpha_1,\alpha_2,\ldots,\alpha_p;\,\rho_1,\,\rho_2,\ldots,\,\rho_q;\,x)
$$

$$
=\frac{2}{\Gamma(\alpha_1-a)}\sum_{r=0}^{\infty}\frac{\Gamma(a+r)\Gamma(\tfrac{1}{2}a+1+r)}{\Gamma(r+1)\Gamma(\tfrac{1}{2}a+r)}\,x^r
$$

$$
\times\,{}_{p+1}f_{q+1}\left(\begin{matrix}\alpha_1-a,\ \alpha_1+2r,\ \alpha_2+r,\ldots,\ \alpha_p+r\\ a+1+2r,\ \rho_1+r,\ \rho_2+r,\ldots,\rho_q+r\end{matrix};\,-x\right),
$$

where

$$
{}_pf_q(\alpha_1,\ \alpha_2,\ldots,\ \alpha_p;\ \rho_1,\ \rho_2,\ldots,\ \rho_q;\ x)
$$

$$
=\frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\ldots\Gamma(\alpha_p)}{\Gamma(\rho_1)\Gamma(\rho_2)\ldots\Gamma(\rho_q)}\,{}_pF_q(\alpha_1,\ \alpha_2,\ldots,\ \alpha_p;\ \rho_1,\ \rho_2,\ldots,\ \rho_q;\ x).
$$

Putting $x=-1$ and taking $\alpha_2=1+\tfrac{1}{2}\alpha_1$, $\rho_1=\tfrac{1}{2}\alpha_1$, $\rho_2=1+\alpha_1-\alpha_3$, $\rho_3=1+\alpha_1-\alpha_4$, $\rho_4=1+\alpha_1-\alpha_5$, $\rho_5=1+\alpha_1-\alpha_6$, we get

$$
\frac{\Gamma(\alpha_1)\Gamma(1+\tfrac{1}{2}\alpha_1)\Gamma(\alpha_3)\Gamma(\alpha_4)\Gamma(\alpha_5)\Gamma(\alpha_6)}{\Gamma(\tfrac{1}{2}\alpha_1)\Gamma(1+\alpha_1-\alpha_3)\Gamma(1+\alpha_1-\alpha_4)\Gamma(1+\alpha_1-\alpha_5)\Gamma(1+\alpha_1-\alpha_6)}
$$

$$
\times\,{}_6F_5\left(\begin{matrix}\alpha_1,\ 1+\tfrac{1}{2}\alpha_1,\ \alpha_3,\ \alpha_4,\ \alpha_5,\ \alpha_6\\ \tfrac{1}{2}\alpha_1,\ 1+\alpha_1-\alpha_3,\ 1+\alpha_1-\alpha_4,\ 1+\alpha_1-\alpha_5,\ 1+\alpha_1-\alpha_6\end{matrix};\,-1\right)
$$

$$
=\sum_{r=0}^{\infty}\Bigg[\frac{\Gamma(a+r)\Gamma(\alpha_1+2r)\Gamma(1+\tfrac{1}{2}\alpha_1+r)\Gamma(\alpha_3+r)\Gamma(\alpha_4+r)}{\Gamma(r+1)\Gamma(a+2r)\Gamma(\tfrac{1}{2}\alpha_1+r)\Gamma(1+\alpha_1-\alpha_3+r)\Gamma(1+\alpha_1-\alpha_4+r)}
$$

$$
\times\frac{\Gamma(\alpha_5+r)\Gamma(\alpha_6+r)}{\Gamma(1+\alpha_1-\alpha_5+r)\,\Gamma(1+\alpha_1-\alpha_6+r)}\Bigg]
$$

* *Proc. Edin. Math. Soc.* (2) 6 (1938), 19-23.

$$\times(-1)^r {}_7F_6\left(\begin{matrix}\alpha_1+2r,\ 1+\tfrac{1}{2}\alpha_1+r,\ \alpha_3+r,\ \alpha_4+r,\ \alpha_5+r,\ \alpha_6+r,\ \alpha_1-a \\ \tfrac{1}{2}\alpha_1+r,\ 1+\alpha_1-\alpha_3+r,\ 1+\alpha_1-\alpha_4+r,\ 1+\alpha_1-\alpha_5+r,\ 1+\alpha_1-\alpha_6+r,\ 1+a+2r\end{matrix};1\right).$$

Now let $\alpha_1-a=-m$ where m is a positive integer, and $1+2\alpha_1+m=\alpha_3+\alpha_4+\alpha_5+\alpha_6$. Then summing up ${}_7F_6$ by Dougall's theorem*, we get after a little simplification

$${}_6F_5\left(\begin{matrix}\alpha_1,\ 1+\tfrac{1}{2}\alpha_1,\ \alpha_3,\ \alpha_4,\ \alpha_5,\ \alpha_6 \\ \tfrac{1}{2}\alpha_1,\ 1+\alpha_1-\alpha_3,\ 1+\alpha_1-\alpha_4,\ 1+\alpha_1-\alpha_5,\ 1+\alpha_1-\alpha_6\end{matrix};-1\right)$$

$$=A\,{}_6F_5\left(\begin{matrix}\alpha_1+m,\ 1+\tfrac{1}{2}(\alpha_1+m),\ \alpha_3,\ \alpha_4,\ \alpha_5,\ \alpha_6; \\ \tfrac{1}{2}(\alpha_1+m),\ 1-\alpha_3+\alpha_1+m,\ 1-\alpha_4+\alpha_1+m,\ 1-\alpha_5+\alpha_1+m,\ 1+\alpha_1-\alpha_6+m\end{matrix}\ -1\right),$$

$$(2+2m>\alpha_1)$$

where

$$A=\frac{(1+\alpha_1)_m(1+\alpha_1-\alpha_3-\alpha_4)_m(1+\alpha_1-\alpha_3-\alpha_5)_m(1+\alpha_1-\alpha_4-\alpha_5)_m}{(1+\alpha_1-\alpha_3)_m(1+\alpha_1-\alpha_4)_m(1+\alpha_1-\alpha_5)_m(1+\alpha_1-\alpha_3-\alpha_4-\alpha_5)_m}.$$

Thus we see that a "well-poised" ${}_6F_5(-1)$ is transformed into another "well-poised" ${}_6F_5(-1)$.

Now*

$${}_6F_5\left[\begin{matrix}a,\ 1+\tfrac{1}{2}a,\ b,\ c,\ d,\ e \\ \tfrac{1}{2}a,\ 1+a-b,\ 1+a-c,\ 1+a-d,\ 1+a-e\end{matrix};-1\right]$$

$$=\frac{\Gamma(1+a-d)\Gamma(1+a-e)}{\Gamma(1+a)\Gamma(1+a-d-e)}\,{}_3F_2\left[\begin{matrix}1+a-b-c,\ d,\ e; \\ 1+a-b,\ 1+a-c\end{matrix}\right].$$

We have from the above after some reduction,

$${}_3F_2\left[\begin{matrix}a,\ b,\ c+d+m-1-b; \\ c,\ d\end{matrix}\right]$$

$$=\frac{(c-b)_m(d-b)_m}{(c)_m\,(d)_m}\,{}_3F_2\left[\begin{matrix}a+m,\ b,\ c+d+m-1-b \\ c+m,\ d+m\end{matrix}\right]$$

and

$${}_3F_2\left[\begin{matrix}c,\ b+m,\ a-1; \\ a,\ b\end{matrix}\right]=\frac{\Gamma(a)\Gamma(b)\Gamma(1-c)\Gamma(b+m+1-a)}{\Gamma(b+m)\Gamma(1+b-a)\Gamma(a-c)},$$

$$(1>c+m).$$

Proc. Edin. Math. Soc. 25 (1907), 114—32.

Again Whipple has shown* that if $\frac{1}{2}+\frac{3}{2}a-c-d-e-f=0$, then

$$ {}_6F_5\left[\begin{matrix} a, 1+\frac{1}{2}a, c, d, e, f \\ \frac{1}{2}a, 1+a-c, 1+a-d, 1+a-e, 1+a-f \end{matrix}; 1\right] $$

$$ =\frac{\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-e)\Gamma(1+a-f)\Gamma(\frac{1}{2}-\frac{1}{2}a+c)}{\Gamma(1+a)\Gamma(\frac{1}{2}-\frac{1}{2}a)\Gamma(1+a-c-d)\Gamma(1+a-c-e)\Gamma(1+a-c-f)} \times\frac{\Gamma(\frac{1}{2}-\frac{1}{2}a+d)\Gamma(\frac{1}{2}-\frac{1}{2}a+e)\Gamma(\frac{1}{2}-\frac{1}{2}a+f)}{\Gamma(1+a-d-e)\Gamma(1+a-d-f)\Gamma(1+a-e-f)}; $$

one of the parameters c, d, e, f being a negative integer. Making use of this result and proceeding exactly as before, we find that

$$ {}_5F_4\left(\begin{matrix} \alpha_1, 1+\frac{1}{2}\alpha_1, \alpha_3, \alpha_4, \alpha_5 \\ \frac{1}{2}\alpha_1, 1+\alpha_1-\alpha_3, 1+\alpha_1-\alpha_4, 1+\alpha_1-\alpha_5 \end{matrix}; -1\right) $$

$$ =\frac{\Gamma(1+\alpha_1-\alpha_3)\Gamma(1+\alpha_1-\alpha_4)\Gamma(1+\alpha_1-\alpha_5)\Gamma(\frac{1}{2}-\frac{1}{2}\alpha_1+\alpha_3)}{\Gamma(1+\alpha_1)\Gamma(\frac{1}{2}-\frac{1}{2}\alpha_1)\Gamma(1+\alpha_1-\alpha_3-\alpha_4)\Gamma(1+\alpha_1-\alpha_3-\alpha_5)} $$

$$ \times\frac{\Gamma(\frac{1}{2}-\frac{1}{2}\alpha_1+\alpha_4)\Gamma(\frac{1}{2}-\frac{1}{2}\alpha_1+\alpha_5)\Gamma(\frac{1}{2}-\frac{1}{2}\alpha_1-m)\Gamma(1+\alpha_1+m)}{\Gamma(1+\alpha_1-\alpha_4-\alpha_5)\Gamma(1-\alpha_3+\alpha_1+m)\Gamma(1-\alpha_4+\alpha_1+m)} \times\frac{1}{\Gamma(1-\alpha_5+\alpha_1+m)} $$

$$ \times{}_6F_5\left[\begin{matrix} \alpha_1+m, 1+\frac{1}{2}(\alpha_1+m), \frac{1}{2}(1+\alpha_1), \alpha_3, \alpha_4, \alpha_5 \\ \frac{1}{2}(\alpha_1+m), \frac{1}{2}(1+\alpha_1)+m, 1-\alpha_3+m+\alpha_1, \\ 1-\alpha_4+m+\alpha_1, 1-\alpha_5+m+\alpha_1, \end{matrix}; -1\right] $$

provided $\frac{1}{2}+\frac{3}{2}\alpha_1+m=\alpha_3+\alpha_4+\alpha_5$.

Thus we see that a "well-poised" ${}_6F_5(-1)$ can be reduced to a "well-poised" ${}_5F_4(-1)$.

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GROUPS IN WHICH THE COMMUTATOR OPERATION SATISFIES CERTAIN ALGEBRAIC CONDITIONS.

BY

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Denote the commutator of two group elements a and b by

$$aba^{-1}b^{-1} = (a, b), \tag{1}$$

then the *associative* law for the commutator-operation is

$$\big((a, b), c\big) = \big(a, (b, c)\big). \tag{2}$$

If (2) is satisfied for every triplet of elements of a group, this group will be called an S-group. If (2) is supposed to be satisfied whenever two of the elements a, b, c are equal (alternating law), then the group will be called an L-group. Hence every S-group is an L-group. It will be proved that a group is an S-group if and only if its commutator-group is situated in the centre (§2). The L-groups which can be generated by two elements are necessarily S-groups; these groups can be represented in a very simple manner (§ 3). In general, L-groups are not S-groups. The second commutators $s = \big((a, b), c\big)$ of an L-group form a sub-group every element of which satisfies $s^3 = 1$ and is commutative with every element of the L-group (§ 6). In an S-group the commutator-operation is distributive with the group-composition, i.e.

$$\begin{aligned}(ab, c) &= (a, c)(b, c)\\ (c, ab) &= (c, a)(c, b).\end{aligned} \tag{3}$$

On the other hand each of the formulae (3) implies that the group is an S-group.

1. Suppose that (2) holds for $b=c$, then

$$\big((a, b), b\big) = 1. \tag{4}$$

This equation is equivalent to $b^{-1}(a, b)b = (a, b)$, and here the left side is equal to (b^{-1}, a). Therefore the condition (4) is equivalent to $(a, b) = (b^{-1}, a)$, and since this condition holds for every pair of elements a, b, b^{-1}, a etc. we get

$$\left.\begin{array}{r}(a, b) = (b^{-1}, a) = (a^{-1}, b^{-1}) = (b, a^{-1})\\ (a, b)^{-1} = (b, a) = (a, b^{-1}) = (b^{-1}, a) = (a^{-1}, b)\end{array}\right\}. \tag{5}$$

From (4) it follows that $\big(b, (a, b)\big) = 1$; moreover b is commutative with $(a, b)^{-1} = (b, a)$, hence $\big((b, a), b\big) = 1$. Hence (4) is a necessary and sufficient condition for an L-group. Since in an L-group, (a, b) is commutative with a and with b, the following formulae can be obtained easily

$$(a^m, b) = a^{m-1}(a, b)ba^{1-m}b = (a, b)(a^{m-1}, b);$$

whence by mathematical induction, it follows for positive integral m, from formula (5), that

$$(a^m, b) = (a, b)^m = (b^{-1}, a)^m = (b^{-m}, a) = (a, b^m).$$

Moreover $\quad (a, b)^{-m} = (b, a)^m = (b, a^m) = (a^{-m}, b).$

Hence for arbitrary integral numbers p and q

$$(a^p, b^q) = (a, b)^{pq}. \tag{6}$$

Moreover $(a, a^p b) = a^{p+1}ba^{-1}b^{-1}a^{-p} = a^{p+1}(b, a^{-1})a^{-(p+1)}$, hence

$$(a, a^p b) = (a, b). \tag{7}$$

Similarly

$$\big((b, c), ab\big) = (b, c)ab(b^{-1}c)b^{-1}a^{-1} = \big((b, c), a\big). \tag{8}$$

2. Consider now an S-group. Then;

$$\big((a, b), c\big) = \big(a, (b, c)\big).$$

Since every S-group is an L-group, the left side can be represented by $(a, b)c(a, b^{-1})\, c^{-1}$ and the right side by $a(c, b^{-1})\, a^{-1}(b^{-1}c)$. Therefore $ba^{-1}b^{-1}cab^{-1}a^{-1}c^{-1}bab^{-1}cbc^{-1} = 1$. Put $b^{-1}c = d$, then

$$(d^{-1}, a^{-1})b^{-1}(a^{-1}, d^{-1})\, b = 1, \text{ i.e. } \big(b, (d, a)\big) = 1. \quad (9)$$

Since a, b, c are arbitrary elements of the group, a, b, d are also arbitrary elements. Hence the second commutators in the group are all equal to 1, i.e. the commutator-group is situated in the centre. On the other hand this condition implies that $\big(a, (b, c)\big) = 1 = \big((a, b), c\big)$ for every triplet of elements of the group. Hence, *A group is an S-group if and only if its commutator-group is situated in the centre.*

To prove that in an S-group the distributive law

$$(ab, c) = (a, c)(b, c)$$

holds, consider $(ab, c)(c, b)(c, a) = ac\big((c^{-1}, b), a^{-1}\big)c^{-1}a = 1$, which proves the formula. Moreover

$$(c, ab) = (ab, c)^{-1} = (c, b)(c, a) = (c, a)(c, b),$$

since the commutators are commutative in an S-group. Hence both the distributive laws hold. The argument is reversible. If any one of the distributive laws holds, the group is an S-group and therefore the other distributive law holds too.

3. Let L_2 be an L-group which is generated by two elements, say a and b. Then (a, b) is commutative with a and with b; hence it is commutative with every element of L_2. The commutator (a, b) is therefore identical with all its conjugates. Thus the commutator-group $G(L_2)$ is generated by (a, b) alone, and it belongs to the centre of L_2. Hence L_2 is an S-group, and every element of L_2 can be represented by

$$a^p b^q (a, b)^r = [p, q, r] = s. \quad (10)$$

Let $[p', q', r'] = s'$, $[p'', q'', r''] = s''$, then

$$ss' = [p + p', q + q', r + r' - p'q]; \quad (11)$$

and in consequence of (11),

$$s^{-1} = [-p, -q, -r-pq];\ s^n = [np, nq, nr - \tbinom{n}{2}pq]$$

$$(s, s') = \left[0, 0, \begin{vmatrix} p & q \\ p' & q' \end{vmatrix}\right];\ \Big((s, s'), s''\Big) = [0, 0, 0]$$

$$(ss')s'' = [p+p'+p'', q+q'+q'', r+r'+r''-p'q-p''q-p''q'] \\ = s(s's''). \qquad (12)$$

If p, q, r run independently over all the integral numbers, and the composition of the symbols (10) is defined by (11), then (12) holds and the elements (10) form an S-group generated by the two elements $[1, 0, 0] = a$ and $[0, 1, 0] = b$. This group will be denoted by G_0. On the other hand every L_2 is homomorphic to G_0. In particular, if p and q run over all the integrals, whereas r is taken mod. an integral number k, then we get a group G_k in which the commutator-group is of order k. G_1 is abelian, and for $k = 0$, we obtain the group denoted already by G_0.

If L_2 is homomorphic, but not isomorphic to G_0, then L_2 is homomorphic to G_k, where $k > 0$.

PROOF. Represent L_2 by (10). Since L_2 is not isomorphic to G_0, there must be a triplet $p, q, r \neq 0, 0, 0$ for which $[p, q, r] = [0, 0, 0]$. If $p = q = 0$, $r \neq 0$, then $(a, b)^r = 1$, and L_2 is homomorphic to G_r. Suppose $p, q \neq 0, 0$; then $a^p b^q = (a, b^{-r})$ belongs to the centre of L_2. Hence $a^p b^q a^{-p} = b^q$, thus $(a^p, b^q) = 1$. Since $(a^p, b^q) = (a, b)^{pq}$, the group L_2 is homomorphic to G_{pq}.

Investigate now the group G_k. An element s belongs to the centre if and only if $(s, s') = 1$ for every element s', that means [see (12)], if and only if $pq' - qp' \equiv 0 \pmod{k}$ for every pair p, q of integers, i.e. when p and q are both divisible by k. The centre of G_k consists therefore of the elements $[kp, kq, r]$, and it is the direct product of two infinite cyclic groups generated by $[k, 0, 0]$ and $[0, k, 0]$ respectively and a cycle of order k generated by $[0, 0, 1]$. The kth powers of the elements of G_k

$$[p, q, r]^k = [kp, kq, \tbinom{k}{2}\, pq]$$

lie in the centre. If k is odd, they run over $[kp, kq, 0]$; if k is even, they take the values $[pk, qk, 0]$ for pq even and $[pk, qk, k/2]$ for pq odd.

Suppose L_2 is homomorphic, but not isomorphic to G_0; thus the order of the commutator-group is finite, say k; then L_2 is homomorphic to G_k. Suppose L_2 not to be isomorphic to G_k, then there exist $p, q \neq 0, 0$, and r such that $[p, q, r] = [0, 0, 0]$. Thus $a^p b^q = (a, b)^{-r}$ lies in the centre, and therefore $(a, a^p b^q) = 1$, but the left side is equal to $(a, b^q) = (a, b)^q$ [see (7) and (6)]. Since k is the order of (a, b), it follows that q is divisible by k; similarly p is divisible by k. Hence we get all the groups L_2 in which the commutator-group is of order k if we consider relations in G_k of the type

$$[kc_i, kd_i, e_i] = 1; \tag{13}$$

the symbols on the left side represent elements of the centre of G_k. Hence the system of relations (13) implies

$$[k\Sigma t_i c_i, k\Sigma t_i d_i, \Sigma t_i e_i] = 1, \tag{14}$$

where the summation is taken over any finite number of relations, and the t's are arbitrary integers. These can be chosen in such a way that $\Sigma t_i c_i = c$ is the H.C.F. of all the c's; hence $c_i = m_i c$. Let (13) be a (not necessarily finite) system of generating relations. Put

$$\Sigma t_i d_i = d, \quad \Sigma t_i e_i = e, \quad d_i - m_i d = d_i', \quad e_i - m_i e = e'_i,$$

then $$[kc, kd, e] = 1, \quad [0, kd_i', e_i'] = 1 \tag{13'}$$

form another system of generating relations. Let $d' = \Sigma s_j d'_j$ be the H.C.F. of all the d_i', and $e' = \Sigma s_j e_j'$, $d_i' = n_i d'$, $e''_i = e'_i - n_i e'$, then we get a new system of generating relations:

$$[kc, kd, e] = 1 = [0, kd', e'] = [0, 0, e_i''].$$

Now k is the order of (a, b); hence every e_i'' is divisible by k and can therefore be replaced by 0. Thus the relations can be generated by not more than two of them,

$$[kc, kd, e] = 1, \quad [0, kd', e'] = 1. \tag{13''}$$

Since a^k, b^k and (a, b) are commutative, we can apply the theory of elementary divisors for the relations between them. The matrix $\begin{pmatrix} c & d \\ 0 & d' \end{pmatrix}$ is congruent to a matrix $\begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix}$ where e_1 is a divisor of e_2. Hence the relations (13″) can be replaced by

$$(a^{kp_1}b^{kq_1})^{e_1} = (a, b)^{g_1}, \quad (a^{kp_2}b^{kq_2})^{e_2} = (a, b)^{g_2},$$

where

$$\begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix} = 1.$$

Put $a^{p_i}b^{q_i} = a_i$ for $i = 1, 2$; then a_1, a_2 are a system of elements generating L_2 and $(a_1, a_2) = (a, b)$ [see (12)].

$$a_i^k = a^{p_ik}b^{q_ik}\,(a, b)^{\binom{k}{2}p_iq_i}.$$

Hence the two relations $a_i^{ke_i} = (a_1, a_2)^{f_i}$ are equivalent to the relations (13). Finally put $e_1 = m$, $e_2 = mn$, then we get L_2 from G_0 by the relations

$$\left.\begin{aligned} &(a_1, a_2)^k = 1 \\ &a_1^{km} = (a_1, a_2)^{f_1}, \text{ where } f_1 = 0 \text{ for } km = 0 \\ &a_2^{kmn} = (a_1, a_2)^{f_2}, \text{ where } f_2 = 0 \text{ for } kmn = 0. \end{aligned}\right\} \qquad (15)$$

These formulae include also the case $m = f_1 = f_2 = 0$, where $L_2 = G_k$, and the case $n = f_2 = 0$, where only one relation is imposed on G_k. If $k \neq 0$, $m \neq 0$, $n \neq 0$, every element of L_2 is of finite order.

4. Consider now L-groups generated by more than two elements. For abbreviation denote

$$xyzx^{-1}y^{-1}z^{-1} = (x, y, z).$$

Let a, b, c be any three elements of an L-group. Denote $(b, c) = A$, $(c, a) = B$, $(a, b) = C$, $(a, b, c) = D$. a is commutative with B and C, b with C and A, c with A and B. Now $(ab, cb) = D$. Hence D is commutative with ab and cb. Thus

$$\begin{aligned} D &= (ab)^{-1}Dab = (c, a^{-1}, b^{-1}) \\ &= (cb)^{-1}Dcb = (b^{-1}, c^{-1}, a). \end{aligned}$$

The L-group generated by a, b, c can also be generated by other triplets of elements. Some of these can be obtained by the transformations S, S^2, T and R determined by the first three columns of the following scheme; the transformation of A, B, C, D is then uniquely determined as given there; the significance of E will be explained later.

	a	b	c	A	B	C	D	E
S	c	a^{-1}	b^{-1}	C	A^{-1}	B^{-1}	D	E
S^2	b^{-1}	c^{-1}	a	B^{-1}	C^{-1}	A	D	E
T	cb	b^{-1}	ab	C	D	A	B	E
R	c	b	a	C^{-1}	B^{-1}	A^{-1}	D^{-1}	E^{-1}.

From every formula which holds in every L-group for arbitrary elements a, b, c, we obtain more such formulae by the transformations S, S^2, T, R.

Now $BD = a^{-1}cac^{-1}.\ ca^{-1}b^{-1}c^{-1}ab$

$= a^{-1}.\ cb^{-1}c^{-1}b.\ a.\ a^{-1}b^{-1}ab = a^{-1}AaC$

$= (a^{-1}, A)AC = (A, a)AC.$

Hence $(A, a) = BDC^{-1}A^{-1}$. Hence by S^2 and S,

$(B, b) = C^{-1}DA^{-1}B.$

$(C, c) = A^{-1}DBC^{-1}$, transform with T

$(A, ab) = C^{-1}BDA^{-1}$, but $(A, ab) = \big((b, c), ab\big)$

$= (A, a)$ as shown by (8). Hence

$BDC^{-1} = C^{-1}BD$, thus $DC^{-1}D^{-1}C = B^{-1}C^{-1}BC$, i.e.

$(C, D) = (B, C)$, by T:

$(A, B) = (D, A)$, by S

$(A, C) = (D, C)$, and hence by T and by S

$(C, A) = (B, A)$

$(B, C) = (B, D)$. Hence

$(D, A) = (A, B) = (A, C) = (D, C) = (C, B)$

$= (D, B) = \Gamma$, and by R

$(D, C) = (C, B) = (C, A) = (D, A) = (A, B)$

$= (D, B).$

Hence $(A, C) = (C, A)$, thus $\Gamma^2 = 1$ and as Γ is a commutator of A, B, C and D, it is commutative with these four elements .For the same reason, (A, a) is commutative with A. Hence $BDC^{-1}A^{-1} = A^{-1}BDC^{-1}$, but we get the right side from the left by three transpositions. Each of these transpositions creates a factor $\Gamma = \Gamma^{-1}$ which is commutative with A, B, C and D; in this way we get the right side with an additional factor $\Gamma^3 = \Gamma$. Hence $\Gamma = 1$. The four elements A, B, C and D of the commutator-group are therefore commutative. Hence

$$(A, a) = (B, b) = (C, c) = A^{-1}BC^{-1}D = E. \qquad (17)$$

The element E defined by (17) is a commutator of a, b, and c; it is therefore commutative with these three elements and with the L-group generated by them. E is invariant for S, S^2 and T and is transformed into E^{-1} by R [as anticipated by the last column of (16)]. When calculating with a, b, c, A, B, C, D, E, at every interchange of a, b, c, a factor A or B or C must be inserted, whereas when A is interchanged with a, or B with b, or C with c, a factor E is created which can be put at the end of the formula as it is commutative with all the elements. More precisely, when $A^{\varepsilon} a^{\delta}$ is changed to $a^{\delta} A^{\varepsilon}$ (where δ and ε are ± 1), then a factor $E^{\varepsilon\delta}$ is created. Now we express the transformation of D by R in two different ways:

$$D \to D^{-1} = A^{-1}BC^{-1}E^{-1}$$

or

$$D = b^{-1}c^{-1}abca^{-1} \to b^{-1}a^{-1}cbac^{-1} = b^{-1}BcC^{-1}bc^{-1}$$
$$= BC^{-1}b^{-1}cbc^{-1}E^2 = A^{-1}BC^{-1}E^2.$$

Hence $E^3 = 1$. (1)

5. Consider the L-group generated by a, b, c. Since A is commutative with b and c, and $a^{-1}Aa = AE$, $aAa^{-1} = AE^{-1}$, the conjugates of A can be represented by A and E, and therefore the commutator-group of the L-group is generated by the elements A, B, C, E which

are commutative with each other. The elements of the group itself can therefore be represented by

$$\alpha = a^p b^q c^r A^P B^Q C^R E^e = [p, q, r | P, Q, R | e], \qquad (19)$$

where e runs over the classes mod 3. Let

$$\beta = [p', q', r' | P', Q', R' | e']; \gamma = [p'', q'', r'' | P'', Q'', R'' | e''];$$

then the relations obtained in §4 furnish

$$\alpha\beta = [p+p', q+q', r+r' | P+P'-q'r, Q+Q'+p'r, R+R'-p'q | e+e'+\Sigma p'P+p'q'r-p'qr-p'qr']$$

$$\alpha^{-1} = [-p, -q-r | -P-qr, -Q+pr, -R-pr | -e+\Sigma pP-pqr]$$

$$(\alpha, \beta) = \left[0, 0, 0 \,\middle|\, \begin{vmatrix} q & r \\ q' & r' \end{vmatrix}, \begin{vmatrix} r & p \\ r' & p' \end{vmatrix}, \begin{vmatrix} p & q \\ p' & q' \end{vmatrix} \,\middle|\, \Sigma p'P - \Sigma pP' + (p'+p)(q'-q)(r'+r) + pqr - p'q'r' \right]$$

$$[(\alpha, \beta), \gamma] = \left[0, 0, 0 \mid 0, 0, 0 \,\middle|\, \begin{vmatrix} p & q & r \\ p' & q' & r' \\ p'' & q'' & r'' \end{vmatrix} \right]. \qquad (20)$$

In these formulae, Σ runs always over the three characters p, q, r (capital and small respectively). The products $(\alpha\beta)\gamma$ and $\alpha(\beta\gamma)$ show a difference of $3p''q'r$ in the last coefficient; since this coefficient is determined mod 3 only, $(\alpha\beta)\gamma = \alpha(\beta\gamma)$. Hence if in (19) the coefficients p, q, r, P, Q, R run independently over all the integral numbers, whereas e runs over the three classes of residues mod 3, and the composition is determined by the first formula (20), then we get a group L_3 which is an L-group and every L-group generated by three elements is homomorphic to L_3. The last formula shows that the second commutator-group consists of three different elements. Hence L_3 is not an S-group. The same formula shows also that

$$\left.\begin{aligned} \delta &= \big((\alpha, \beta), \gamma\big) = \big((\beta, \gamma), \alpha\big) = \big((\gamma, \alpha), \beta\big) \\ \delta^{-1} &= \big(\gamma, (\alpha, \beta)\big) = \big(\alpha, (\beta, \gamma)\big) = \big(\beta, (\gamma, \alpha)\big) \\ \delta^3 &= 1. \end{aligned}\right\} \qquad (21)$$

The second commutator of three elements depends only on the first three coordinates, whereas the commu-

tativity of two elements depends on the first six co-ordinates.

The group L_3 can be generalised in the following way. Let Δ be an integral domain, Σ any extension of Δ. If p, q, r run over Δ, whereas P, Q, R run over Σ, and e runs over the cosets of Σ mod 3Δ, then (19) and (20) define again an L-group.

6. Let L be an arbitrary L-group. Every triplet of elements of L generates a group which is homomorphic to L_3. Hence for every triplet of elements of L, formula (21) is satisfied.

Put $\alpha = \alpha_1, \beta = \alpha_2, \gamma = (\beta_1, \beta_2)$, then

$$\lambda = \big((\alpha_1, \alpha_2), \ (\beta_1, \beta_2) \big) = \big((\alpha_2, \ (\beta_1, \beta_2) \big), \alpha_1)$$

$$= \big(\alpha_1, ((\beta_1, \beta_2), \alpha_2 \big) = \big(\alpha_1, ((\beta_2, \alpha_2), \beta_1 \big) = \big(\alpha_1, (\beta_1, (\alpha_2, \beta_2)) \big).$$

Put $\alpha = (\alpha_1, \alpha_2), \beta = \beta_1, \gamma = \beta_2$, then

$$\lambda = \big(\beta_1, (\beta_2, (\alpha_1, \alpha_2)) \big) = \big(\beta_1, ((\alpha_2, \alpha_1), \beta_2) \big)$$

$$= \big(\beta_1, ((\beta_2, \alpha_2), \alpha_1) \big) = \big(\alpha_1, (\beta_1, (\beta_2, \alpha_2)) \big) = \lambda^{-1};$$

but since λ is a second commutator, $\lambda^3 = 1$. Hence $\lambda^2 = \lambda^3 = 1$, thus $\lambda = 1$.

Every arbitrary element α_1 is therefore commutative with every second commutator $\big(\beta_1, (\alpha_2, \beta_2) \big)$. Thus the group generated by the second commutators is situated in the centre of L.

It will be proved now that *in an L-group the elements s satisfying $s^3 = 1$ form a group.* This statement implies that the elements of the second commutator-group are all of order 3 or 1.

PROOF. Let $a^3 = b^3 = 1$, then $(ab)^3 = aba.(b, a)ab^2 = aba^2b^2(b, a) = (a, b)(b, a) = 1$. Hence if a sub-group of L is generated by elements of order 3, every element satisfies $s^3 = 1$. The converse of this statement is also true: *If in a group T every element satisfies $s^3 = 1$, then T is an L-group.*

PROOF. Let a and b be elements of T, then

$$1 = (ab)^3 = (a, b)(b, a^2)(a^2, b^2) \equiv M$$

and

$$1 = (ab^2)^3 = (a, b^2)(b^2, a^2)(a^2, b) \equiv (b^2, a^2)(a^2, b)(a, b^2) \equiv N;$$

therefore $1 = MN = (a, b)(a, b^2)$. Hence $(a, b) = (b^{-1}, a)$ for every pair of elements a, b of T. Hence T is an L-group.

If L is generated by n elements $a_1, \ldots, a_n$, we can represent the elements of L by

$$a_1^{p_1} \ldots a_n^{p_n}(a_1, a_2)^{q_1} \ldots (a_{n-1}, a_n)^{q_s}\Big(a_1, (a_2, a_3)^{r_1}\Big) \ldots \Big((a_{n-2}, (a_{n-1}, a_n)^{r_t}\Big),$$

where $s = \binom{n}{2}$, $t = \binom{n}{3}$, and $r_1, \ldots, r_t$ take the values 0, 1, 2 only. If in particular the group is of the type T, then the symbols p and q also take only 0, 1, 2. Hence the order of T is at most 3^{n+s+t}. This particular result has been proved already in an earlier paper*.

* F. Levi and B. L. van der Waerden: 'Uber eine besondere Klasse von Gruppen' *Abh. Math. Sem. Hamburg Univ.* 9, 154.

Errata in the paper "On the number of generators of a free product, and a lemma of Alexander Kurosch", *Jour. Indian Math. Soc.* (2) 5, 149-55.

p. 150. formula (3). For $m(A*B) \leqslant m(B)$ read $m(A*B) \geqslant m(B)$.

p. 154. last line of the theorem. For *subgroup of G* read *subgroup of A*.

A CLASS OF INFINITE INTEGRALS II

BY

B. MOHAN, *Benares*

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1. In a recent paper* I have studied the properties of the function

$$\psi_\nu(x) = \sum_0^\infty \frac{\Gamma(\frac{1}{2}+r)\, x^{\nu+2r}}{2^{\nu+2r} r!\, \Gamma(\nu+r+1)}. \tag{1.1}$$

I have shown therein that

$$\psi_\nu(x) = \frac{(\frac{1}{2}x)^\nu}{\Gamma(\nu+\frac{1}{2})} \int_0^1 \frac{(1-u)^{\nu-\frac{1}{2}}}{\sqrt{u}}\, e^{\frac{1}{4}x^2 u} du,\ R(\nu) > -\tfrac{1}{2} \tag{1.2}$$

and $\psi_\nu(x) \sim x^{-\nu-1}\, e^{\frac{1}{4}x^2}$ for large x and $R(\nu) > -\frac{1}{2}$.

The object of this note is to study a few infinite integrals involving this function.

2. We proceed to evaluate the integral

$$I = \int_0^\infty x^{-p} e^{-\frac{1}{4}c^2x^2} J_m(bx)\, \psi_{m+p+1}(cx)\, dx,$$

which is convergent, for $R(m) > -1$, $R(m+p) > -\frac{3}{2}$, $R(m+2p) > -\frac{5}{2}$; $R(b) > 0$, $R(c) > 0$.

We have, by (1.2),

$$I = \int_0^\infty x^{-p} e^{-\frac{1}{4}c^2x^2} J_m(bx)\, \frac{(\frac{1}{2}cx)^{m+p+1} dx}{\Gamma(m+p+\frac{3}{2})} \int_0^1 \frac{(1-u)^{m+p+\frac{1}{2}}}{\sqrt{u}} \times e^{\frac{1}{4}c^2x^2u}\, du$$

$$= \frac{c^{m+p+1}}{2^{m+p+1}\,\Gamma(m+p+\frac{3}{2})} \int_0^1 \frac{(1-u)^{m+p+\frac{1}{2}}}{\sqrt{u}}\, du \int_0^\infty x^{m+1} e^{-\frac{1}{4}c^2x^2(1-u)} \times J_m(bx)\, dx,$$

*Properties of a certain Confluent Hypergeometric Function—*Bull. Cal. Math. Soc.* 33 (1941), 99—103.

a process easily justifiable. Thus, using Sonine's formula,* we have

$$I = \frac{b^m c^{p-m-1}}{2^p \Gamma(m+p+\frac{3}{2})} \int_0^1 \frac{(1-u)^{p-\frac{1}{2}}}{\sqrt{u}} e^{\frac{-b^2}{c^2(1-u)}} du.$$

Putting $u = \frac{t}{1+t}$, $du = \frac{dt}{(1+t)^2}$, we get

$$I = \frac{b^m c^{p-m-1}}{2^p \Gamma(p+m+\frac{3}{2})} e^{-b^2/c^2} \int_0^\infty \frac{t^{-\frac{1}{2}} e^{-b^2 t/c^2}}{(1+t)^{p+1}} dt.$$

On using a formula given by Whittaker† we have

$$\int_0^\infty x^{-p} e^{-c^2x^2/4} J_m(bx)\, \psi_{m+p+1}(cx)\, dx$$

$$= \frac{\pi^{\frac{1}{2}} b^{m+p-\frac{1}{2}} e^{-b^2/2c^2}}{2^p c^{m+\frac{1}{2}} \Gamma(p+m+\frac{3}{2})} W_{-\frac{1}{2}p-\frac{1}{4}, -\frac{1}{2}p-\frac{1}{4}} \left(\frac{b^2}{c^2}\right), \qquad (2.1)$$

where $R(m) > -1$, $R(m+p) > -\frac{3}{2}$, $R(m+2p) > -\frac{5}{2}$;

$R(b) > 0$, $R(c) > 0$.

Particular Cases

(i) $p = -1$. On using the formula

$$W_{m+\frac{1}{2}, \pm m}(z) = z^{m+\frac{1}{2}} e^{-z/2},$$

we get

$$\int_0^\infty x e^{-\frac{1}{4}c^2x^2} J_m(bx)\, \psi_m(cx)\, dx = \frac{2\pi^{\frac{1}{2}} b^{m-1}}{c^{m+1} \Gamma(m+\frac{1}{2})} e^{-\frac{b^2}{c^2}}, R(m) > -\tfrac{1}{2}.$$

For $m = \frac{1}{2}$ this formula becomes

$$\int_0^\infty x^{\frac{1}{2}} e^{-c^2x^2/4} \sin bx\, \psi_{\frac{1}{2}}(cx)\, dx = \frac{\pi\sqrt{2}}{c^{\frac{3}{2}}} e^{-\frac{b^2}{c^2}}.$$

(ii) $p = 0$. On using the formula

$$W_{k, \pm\frac{1}{4}}(x) = 2^{\frac{1}{4}-k} x^{\frac{1}{4}} D_{2k-\frac{1}{2}}(\sqrt{2x}),$$

*G. N. Watson: *Theory of Bessel Functions*, § 13.3 (4).

† Whittaker and Watson: *Modern Analysis*, § 16.12.

we get

$$\int_0^\infty e^{-c^2x^2/4} J_m(bx)\, \psi_{m+1}\,(cx)\,dx$$

$$= \frac{(2\pi)^{\frac{1}{2}} b^m}{c^{m+1}\,\Gamma(m+\frac{3}{2})}\, e^{-\frac{b^2}{2c^2}}\, D_{-1}\left(\frac{b}{c}\sqrt{2}\right),\ R(m) > -1.$$

For $m = \pm\frac{1}{2}$ this formula assumes the following forms

$$\int_0^\infty x^{-\frac{1}{2}}\, e^{-c^2x^2/4}\, \psi_{\frac{3}{2}}(cx) \sin bx\, dx = \pi b\, c^{-\frac{3}{2}} e^{-\frac{b^2}{2c^2}} D_{-1}\left(\frac{b}{c}\sqrt{2}\right)$$

$$\int_0^\infty x^{-\frac{1}{2}}\, e^{-c^2x^2/4}\, \psi_{\frac{1}{2}}\,(cx) \cos bx\, dx = \frac{\pi}{\sqrt{c}} e^{-\frac{b^2}{2c^2}} D_{-1}\left(\frac{b}{c}\sqrt{2}\right).$$

(iii) $p = \frac{1}{2}$. On using the formula

$$W_{n,\frac{1}{2}}(2z) = \Gamma(n+1)\, k_{2n}(iz),$$

we get

$$\int_0^\infty x^{-\frac{1}{2}}\, e^{-c^2x^2/4}\, J_m(bx)\psi_{m+\frac{3}{2}}\,(cx)\,dx$$

$$= \frac{\pi b^m\, e^{-\frac{b^2}{2c^2}}}{\sqrt{2}\, c^{m+\frac{1}{2}}\Gamma(m+2)}\, k_{-1}\left(\frac{b^2}{2c^2}\right),\ R(m) > -1.$$

For $m = \pm\frac{1}{2}$, this formula becomes

$$\int_0^\infty \frac{\sin bx}{x} e^{-c^2x^2/4}\, \psi_2(cx)\,dx = \frac{2\pi b}{3c}\, e^{-\frac{b^2}{2c^2}}\, k_{-1}\left(\frac{b^2}{2c^2}\right),$$

$$\int_0^\infty \frac{\cos bx}{x} e^{-c^2x^2/4}\, \psi_1(cx)\,dx = \pi e^{-\frac{b^2}{2c^2}}\, k_{-1}\left(\frac{b^2}{2c^2}\right).$$

(iv) $p = -\frac{1}{2}$. On using the formula

$$W_{0,\nu}\,(2z) = \left(\frac{2z}{\pi}\right)^{\frac{1}{2}} K_\nu(z),$$

we get

$$\int_0^\infty x^{\frac{1}{2}}\, e^{-c^2x^2/4}\, J_m(bx)\psi_{m+\frac{1}{2}}\,(cx)\,dx$$

$$= \frac{2^{\frac{1}{2}} b^m}{c^{m+\frac{3}{2}}\,\Gamma(m+1)}\, e^{-\frac{b^2}{2c^2}} K_0\left(\frac{b^2}{2c^2}\right),\ R(m) > -1.$$

For $m = \pm\frac{1}{2}$, we have

$$\int_0^\infty e^{-c^2x^2/4}\,\psi_1(cx)\sin bx\,dx = \frac{2b}{c^2}\,e^{-\frac{b^2}{2c^2}}K_0\left(\frac{b^2}{2c^2}\right),$$

$$\int_0^\infty e^{-c^2x^2/4}\,\psi_0(cx)\cos bx\,dx = \frac{1}{c}\,e^{-\frac{b^2}{2c^2}}K_0\left(\frac{b^2}{2c^2}\right).$$

(v) $m = \frac{1}{2}$.

$$\int_0^\infty x^{-p-\frac{1}{2}}\,e^{-c^2x^2/4}\,\psi_{p+\frac{3}{2}}(cx)\sin bx\,dx$$

$$= \frac{\pi b^{p+\frac{1}{2}}\,e^{-\frac{b^2}{2c^2}}}{2^{p+\frac{1}{2}}\,c\Gamma(p+2)}\,W_{-\frac{1}{2}p-\frac{1}{4},\,-\frac{1}{2}p-\frac{1}{4}}\left(\frac{b^2}{c^2}\right),\ R(p) > -1.$$

(vi) $m = -\frac{1}{2}$.

$$\int_0^\infty x^{-p-\frac{1}{2}}\,e^{-\frac{1}{4}c^2x^2}\,\psi_{p+\frac{1}{2}}(cx)\cos bx\,dx$$

$$= \frac{\pi b^{p-\frac{1}{2}}}{2^{p+\frac{1}{2}}\,\Gamma(p+1)}\,e^{-\frac{b^2}{2c^2}}\,W_{-\frac{1}{2}p-\frac{1}{4},\,-\frac{1}{2}p-\frac{1}{4}}\left(\frac{b^2}{c^2}\right),\ R(p) > -1.$$

POSTSCRIPT

BY

MISS S. PANKAJAM.

In the review of my first paper 'On the Formal Structure of the Propositional Calculus I' in the *Journal of Symbolic Logic*, the reviewer says:

"The author attempts to compare this system with that of Heyting,.........According to the author this correspondence is obtained by interpreting $\supset$ (the implication of Heyting) as $<$ (the lattice-implication), but this is clearly not the case, in part because the former is an operation and the latter a relation. In order to obtain the correspondence which she desires, it seems to the reviewer necessary to use Birkhoff's Brouwerian Lattice which is obtained by assuming the postulate: *for every two elements, a, b of the lattice, a greatest element* $a \to b$ *exists such that* $a(a \to b) < b$."

It is quite true that in my first paper in which lattice implication was considered only as a relation, a satisfactory identification of lattice implication with Heyting's implication would not have been possible except on very rough lines. The final suggestion of the reviewer amounts to the assertion of the existence of relative product-complements and the identification of $a < b$ with the product-complement of a relative to b, which is precisely what is carried out in the present paper.

A FEW INFINITE INTEGRALS.*

BY

MISS S. SINHA, *Research Scholar, Benares Hindu University*

[Received 25 February 1942.]

1. We consider the following integrals:

$$\int_0^\infty t^{\rho-\mu-\nu-1} \mathcal{J}_\mu(at)\, \mathcal{J}_\nu(bt)\, \mathcal{J}_\rho(ct)\, dt = \frac{2^{\rho-\mu-\nu-1} a^\mu b^\nu \Gamma(\rho)}{c^\rho \Gamma(\mu+1)\, \Gamma(\nu+1)},\dagger$$

$$c>a+b,\ \rho-\mu-\nu<5/2. \qquad (1\cdot 1)$$

$$\int_0^\infty \mathcal{J}_\mu(ct) \prod_{m=1}^n [\mathcal{J}_\nu(a_m t)]\, t^{\mu-n\nu-1}\, dt = \frac{2^\mu \Gamma(\mu)}{c^\mu} \prod_{m=1}^n \left[\frac{(\frac{1}{2}a_m)^\nu}{\Gamma(\nu+1)}\right],\ddagger$$

$$R(n\nu+\tfrac{1}{2}n+\tfrac{1}{2})>R(\mu)>0,\ c>\Sigma a. \qquad (1.2)$$

$$\int_0^\infty \mathcal{J}_{\mu+1}(ct) \frac{\mathcal{J}_\nu\{a(t^2+z^2)^{\frac{1}{2}}\}}{(t^2+z^2)^{\nu/2}}\, t^\mu dt = \frac{2^\mu \Gamma(\mu+1)\, \mathcal{J}_\nu(az)}{c^{\mu+1} z^\nu},$$

$$R(\nu+1)>R(\mu)>-1. \qquad (1.3)$$

We multiply both sides of each of these by $c^{\lambda-1} e^{-p^2c^2}$, integrate with respect to c, and evaluate after inverting the order of integration on the left side, by using Hankel's formula§. We obtain the following results:

$$\int_0^\infty t^{2\rho-\mu-\nu-1} e^{-ct^2} \mathcal{J}_\mu(at)\, \mathcal{J}_\nu(bt)\ {}_1F_1\left(\frac{\rho-\lambda}{2}+1;\ \rho+1;\ ct^2\right) dt$$

$$= \frac{\rho a^\mu b^\nu \Gamma(\frac{1}{2}\lambda-\frac{1}{2}\rho)\{\Gamma(\rho)\}^2}{2^{\mu+\nu+1} c^\rho \Gamma(\mu+1)\, \Gamma(\nu+1)\Gamma(\frac{1}{2}\lambda+\frac{1}{2}\rho)}, \qquad (1.4)$$

$$a+b<0,\ R(\lambda+\rho)>0,\ R(\rho-\mu-\nu)<5/2.$$

$$\int_0^\infty t^{2\mu-n\nu-1} e^{-ct^2} \prod_{m=1}^n \left[\mathcal{J}_\nu(a_m t)\right]\ {}_1F_1\left(\frac{\mu-\lambda}{2}+1;\ \mu+1;\ ct^2\right) dt$$

$$= \frac{\mu[\Gamma(\mu)]^2\, \Gamma(\frac{1}{2}\lambda-\frac{1}{2}\mu)}{c^\mu \Gamma(\frac{1}{2}\lambda+\frac{1}{2}\mu)} \prod_{m=1}^n \left[\frac{(\frac{1}{2}a_m)^\nu}{\Gamma(\nu+1)}\right], \qquad (1.5)$$

$$R(n\nu+\tfrac{1}{2}n+\tfrac{1}{2})>R(\mu)>0;\ R(\mu+\lambda)>0,\ \Sigma a<0.$$

*I wish to thank Dr. B. Mohan for his kind help and interest in the preparation of this note.

†W. N. Bailey, *Proc. L. M. S.* (2), 40 (935), 37-48.

‡G. N. Watson, *Theory of Bessel Functions*, §13.47.

§*Ibid*, § 13.3 (3).

$$\int_0^\infty t^{2\mu+1}\, e^{-ct^2} \frac{J_\nu\{a(t^2+z^2)^{\frac{1}{2}}\}}{(t^2+z^2)^{\nu/2}} \quad {}_1F_1(\tfrac{1}{2}\mu-\tfrac{1}{2}\lambda+\tfrac{3}{2};\ \mu+2;\ ct^2)\,dt$$

$$= \frac{J_\nu(az)\ (\mu+1)\{\Gamma(\mu+1)\}^2\ \Gamma(\frac{1}{2}\lambda-\frac{1}{2}\mu-\frac{1}{2})}{2z^\nu\ c^{\mu+1}\ \Gamma(\frac{1}{2}\mu+\frac{1}{2}\lambda+\frac{1}{2})}, \qquad (1.6)$$

$$R(\nu+1)>R(\mu)>-1,\ R(\lambda+\mu+1)>0.$$

2. If we multiply both sides of (1.1), (1.2), (1.3), by $cJ_\rho(cq)\,e^{-p^2c^2}$, and apply a similar process, and use Weber's formula,* we obtain the following:

$$\int_0^\infty t^{\rho-\mu-\nu-1}\, e^{-\frac{ct^2}{2q}}\ J_\mu(at)\ J_\nu(bt)\ I_\rho(ct)\ dt$$

$$= \frac{1}{2^{\mu+\nu+1}} \frac{a^\mu b^\nu q^\rho}{\rho\Gamma(\nu+1)\Gamma(\mu+1)}\ {}_1F_1\left(\rho;\ \rho+1;\ cq/2\right), \quad (2.1)$$

$$a+b<0,\ R(\rho-\mu-\nu)<\tfrac{5}{2},\ R(\rho)>0,\ |\arg q/c|<\pi/2.$$

$$\int_0^\infty t^{\mu-n\nu-1}\, e^{-\frac{ct^2}{2q}}\ I_\mu(ct) \prod_{m=1}^{n} [J_\nu(a_m t)]\ dt = \frac{1}{2\mu} \prod_{m=1}^{n} \left[\frac{(\frac{1}{2}a_m)^\nu}{\Gamma(\nu+1)}\right]$$

$$\times\ {}_1F_1(2;\ \mu+1;\ cq/2),\ R(n\nu+\tfrac{1}{2}\mu+\tfrac{1}{2})>R(\mu)>0,\ \Sigma a<0,$$

$$|\arg q/c|<\pi/2. \qquad (2.2)$$

$$\int_0^\infty t^\mu\, e^{-\frac{ct^2}{2q}}\ I_{\mu+1}(ct^2) \frac{J_\nu\{a\sqrt{t^2+z^2}\}}{(t^2+z^2)^{\nu/2}}\ dt = \frac{J_\nu(az)(\mu+2)}{2z^\nu}\ q^{\mu+1}$$

$$\times\ {}_1F_1(\mu+1;\ \mu+2;\ cq/2),\ R(\nu+1)>R(\mu)>-1,\ |\arg q/c|<\pi/2. \qquad (2.3)$$

Particular cases of (1.4), (1.5), (1.6) will be obtained by putting $\lambda=1$, and using MacRobert's result†

$$I_n(z) = \frac{e^z}{\Gamma(n+1)} \left(\frac{z}{2}\right)^n {}_1F_1(\tfrac{1}{2}+n;\ 1+2n;\ -2z).$$

Particular cases of (2.1), (2.2), (2.3) will be obtained by putting $\mu=0$, $\nu=0$.

* Watson, *loc. cit.* § 13.31 (1).

† T. M. MacRobert, *Phil. Mag.* 26 (1938), 84.

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